Calculus Review Session

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Introduction

This is designed only as a review of calculus. I assume that you have already been exposed to this material and simply need a quick refresher.

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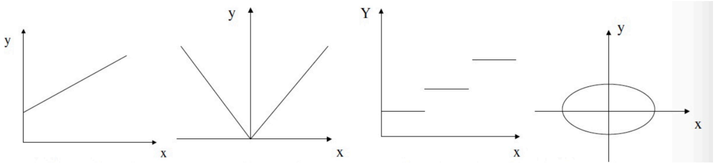
Characteristics of Functions

A **function** is a correspondence that maps each value of x (the independent variable) into a unique value of y (the dependent variable).
A **continuous function** is a function with no break in it (draw a continuous function without lifting your pencil).
A **continuously differentiable function** is a function where the first derivative exists for every value of x.

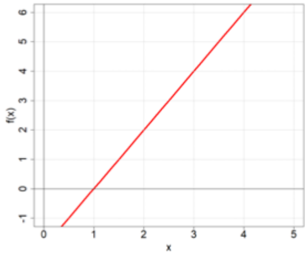


Differentiation in One Variable

The derivative of a function represents the rate of change of the function. For a linear function, the derivative is the slope of the line. For non-linear functions, the derivative will not be constant, but rather will represent the slope of a tangent to the curve at a particular point.

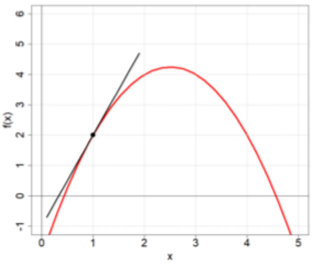
Linear Function

$f(x) = 2x - 2$



Non-Linear Function

$f(x) = -x^2 + 5x - 2$



Power Rule

Power Rule:

$$y = kx^a$$
$$\frac{dy}{dx} = akx^{a-1}$$

Example:

$$y = 15x^3$$
$$\frac{dy}{dx} = ?$$

Chain Rule

Chain Rule:

$$y = f(g(x))$$
$$\frac{dy}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

Example:

$$y = (-3x + 5)^2$$
$$\frac{dy}{dx} = ?$$

Product Rule

Product Rule:

$$y = f(x) \cdot g(x)$$
$$\frac{dy}{dx} = \frac{df}{dx} \cdot g(x) + \frac{dg}{dx} \cdot f(x)$$

Example:

$$y = x^2(x-2)$$
$$\frac{dy}{dx} = ?$$

Quotient Rule

Quotient Rule:

$$y = \frac{f(x)}{g(x)}$$

$$\frac{dy}{dx} = \frac{\frac{df}{dx} \cdot g(x) - \frac{dg}{dx} \cdot f(x)}{g(x)^2}$$

Example:

$$y = \frac{x^2}{(x-2)}$$

$$\frac{dy}{dx} = ?$$

Second, Third and Higher Derivatives

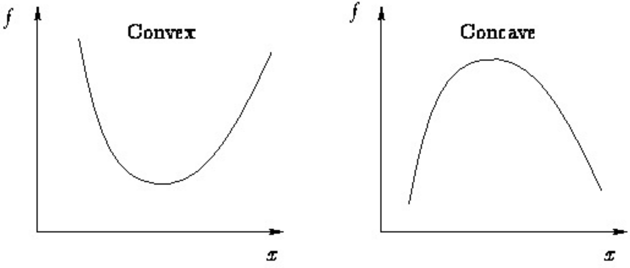
Easy: Differentiate the function again (and again, and again...)
Example:

$$y = x^3 + 3x^2 + 6x + 7$$
$$\frac{dy}{dx} = 3x^2 + 6x + 6$$
$$\frac{d^2y}{dx^2} = ?$$

Interpretation of Second Derivatives

The second derivative tells us about the concavity of the function:

- If $\frac{d^2y}{dx^2} > 0$, the function is concave up (convex).
- If $\frac{d^2y}{dx^2} < 0$, the function is concave down (concave).



Finding Extrema

To find the local maxima and minima of a function, we use the first and second derivatives:

- Find the critical points by setting $\frac{dy}{dx} = 0$.
- Use the second derivative to determine the nature of the critical points.

Example:

$$y = -x^2 + 5x - 2$$
$$\frac{dy}{dx} = -2x + 5$$
$$\frac{dy}{dx} = 0 \Rightarrow x = \frac{5}{2} = 2.5$$
$$\frac{d^2y}{dx^2} = -2 \Rightarrow \text{Maximum at } x = 2.5$$

Inflection Points

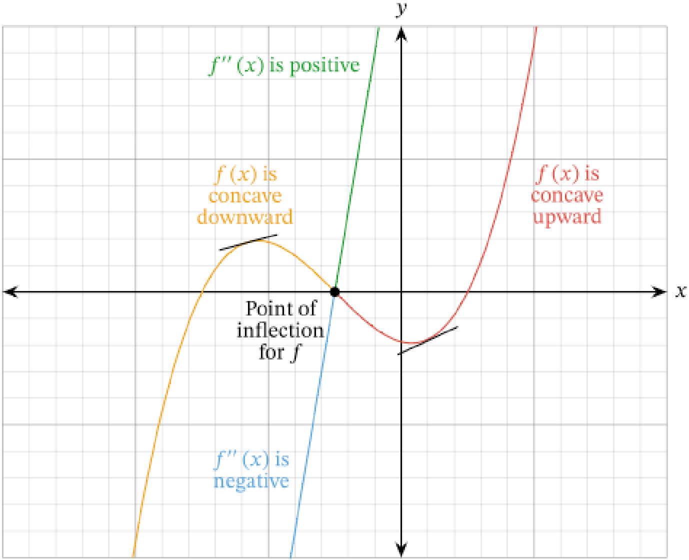
An inflection point is where the function changes from concave to convex, or vice versa. It occurs where the second derivative is zero.

$$\frac{d^2y}{dx^2} = 0$$

Example:

$$y = x^3 - 3x^2 + 4x - 2$$
$$\frac{dy}{dx} = 3x^2 - 6x + 4$$
$$\frac{d^2y}{dx^2} = 6x - 6$$
$$\frac{d^2y}{dx^2} = 0 \Rightarrow x = 1 \text{ (Inflection Point)}$$

Interpreting Graphs of Derivatives



Source: Nagwa[Online]

Rules of Exponents and Logarithms

Rule 1: The natural log (ln) is the inverse of the exponential function e . The two functions essentially cancel each other out.

$$\text{Example:} \quad \ln(e^7) = 7$$

$$\text{Example:} \quad e^{\ln(x)} = x$$

Rule 2: The log of a **product** is the sum of the logs.

$$\text{Example:} \quad \ln(AB) = \ln(A) + \ln(B)$$

$$\text{Example:} \quad \ln(Ae^7) = \ln(A) + 7$$

Rule 3: The log of a **quotient** is the difference of the logs.

$$\text{Example:} \quad \ln\left(\frac{e^2}{c}\right) = \ln(e^2) - \ln(c) = 2 - \ln(c)$$

$$\text{Example:} \quad \ln\left(\frac{e^2}{e^5}\right) = 2 - 5 = -3$$

More Rules of Exponents and Logarithms

Rule 4: The log of a **power** equals the power times the log.

$$\text{Example:} \quad \ln(e^{15}) = 15\ln(e) = 15$$

$$\text{Example:} \quad \ln(uv^a) = \ln(u) + \ln(v^a) = \ln(u) + a\ln(v)$$

Rule 5: Derivatives of $\ln(x)$

$$\frac{d\ln(x)}{dx} = \frac{1}{x}$$

$$\frac{d\ln(2x)}{dx} = \frac{1}{2x} \cdot 2 = \frac{1}{x}$$

Rule 6: Derivatives of e^x

$$\frac{de^x}{dx} = e^x$$

$$\frac{de^{2x}}{dx} = 2e^{2x}$$

Partial Derivatives

The **partial derivative** of a multivariable function, say z = f(x, y), is its derivative with respect to one of the variables, x or y in this case, where the other variables are treated as constants.

Example:

$$z = 3x^2y^3$$
$$\frac{\partial z}{\partial x} = 6xy^3$$
$$\frac{\partial z}{\partial y} = 9x^2y^2$$

More Partial Derivatives

Example:

$$f(x, y, z) = xyz + x^3y + z^8$$
$$\frac{\partial f}{\partial x} = yz + 3x^2y$$
$$\frac{\partial f}{\partial y} = xz + x^3$$
$$\frac{\partial f}{\partial z} = xy + 8z^7$$

Example:

$$z = 4x^2 + y^5$$
$$\frac{\partial z}{\partial x} = ?$$
$$\frac{\partial z}{\partial y} = ?$$

The Total Derivative

The **total derivative** represents the change in a multivariate function with respect to all variables. It is the sum of the partial derivatives of a function for each variable multiplied by the change in that variable. In other words, if you have a function F(x, y) , its total derivative is:

$$\frac{\partial F(x, y)}{\partial x} dx + \frac{\partial F(x, y)}{\partial y} dy$$

Example:

$$z = 3x^2 y^3$$

$$\text{Total Derivative} = 6xy^3 dx + 9x^2 y^2 dy$$

Integration—The Indefinite Integral

Integration is the "reverse" of differentiation. The FUNDAMENTAL THEOREM OF CALCULUS states that the integral of the derivative is the original function plus some constant of integration.

$$\int \frac{df}{dx} dx = f(x) + c$$

Example:

$$f(x) = x^2 + 2$$
$$\frac{df}{dx} = 2x$$
$$f(x) = x^2 + 200$$
$$\frac{df}{dx} = 2x$$
$$\int 2x dx = x^2 + c$$

Rules of Integration

Rule 1: Power Rule

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + c$$

Example:

$$\int x^3 dx = ?$$

Rule 2: Exponential Rule

$$\int e^x dx = e^x + c$$

Example:

$$\int 2e^{2x} dx = ?$$

Rule 3: Logarithmic Rule

$$\int \frac{1}{x} dx = \ln|x| + c$$

Example:

$$\int \frac{2x}{x^2} dx = ?$$

Rule 4: Integrals of Sums

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$$

Example:

$$\int (x^3 + x^2 + 2x)dx = ?$$

Rule 5: Integrals Involving Multiplication

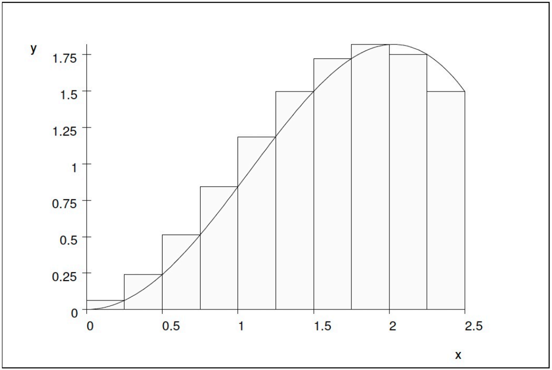
$$\int kf(x)dx = k\int f(x)dx$$

Example:

$$\int 2x^2 dx = ?$$

Integration—The Definite Integral

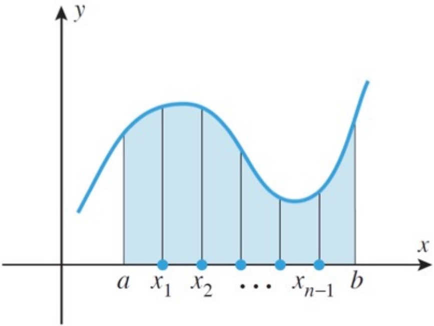
The definite integral represents the area under the curve between two points. The graph below is for the function $x \sin x$. Imagine that we wanted to know the area under that function from 0 to 2.5.We can approximate the area under the curve by dividing the area into a series of rectangles and calculating the area for each rectangle and summing them up.



Riemann Sum

We can approximate the area under the curve by dividing the area into a series of rectangles and calculating the area for each rectangle and summing them up. If you divide the graph into n equal width rectangles, then the area under the curve from values a to b can be given by:

$$\frac{b-a}{n}\sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)$$



Riemann Sum Example

In our example, $n = 10$, $b = 2.5$, $a = 0$. So the Riemann sum representing the area under the curve can be calculated as:

$$
\begin{aligned}
0.25 \sum_{i=1}^{n} f(0.25i) &= 0.25 \sum_{i=1}^{n} (0.25i) \sin(0.25i) \\
&= 0.25(f(0.25) + f(0.5) + f(0.75) + f(1) + \ldots + f(2.5)) \\
&= 2.78
\end{aligned}
$$

Definite Integral

The limit of the Riemann sum when n goes to infinity is the definite integral. It is written as follows:

$$\int_a^b f(x)dx$$

Calculating a Definite Integral

To calculate a definite integral, take the indefinite integral and evaluate that integral at the upper limit of integration (b) and then evaluate the indefinite integral at the lower limit of integration (a) and subtract the second:

$$\int_2^6 3x^2 dx$$

First take the indefinite integral:

$$x^3 + c$$

Evaluate at 6 and forget c:

$$6^3 = 216$$

Evaluate at 2 and forget c:

$$2^3 = 8$$

Subtract latter from former:

$$216 - 8 = 208$$

Comparing Riemann Sum and Definite Integral

How good was our Riemann sum estimate of the integral of $x \sin x$ from 0 to 2.5? That integral is fairly nasty to compute, but software can do it easily. The actual definite integral is:

$$\int_0^{2.5} x \sin x dx = 2.6013$$

and our Riemann sum estimate using 10 rectangles was 2.781.

Exponential Growth and Decay

Imagine that you have $100 to put in the bank and the interest rate (compounded annually) is 3%. If you make no further deposits or withdrawals, how much money will you have in 2 years? In 5 years? Since the interest is compounded annually, after 1 year you have:

$$Y_1 = \$100(1 + 0.03) = \$103$$

After 2 years you have:

$$Y_2 = Y_1(1 + 0.03) = \$100(1 + 0.03)^2 = \$106.09$$

The general formula for calculating your balance with annually compounding interest is:

$$Y_t = Y_0(1 + r)^t$$

So in 5 years you have:

$$Y_5 = 100(1.03)^5 = \$115.93$$

Continuous Compounding

What if the interest compounded more frequently? What if the interest compounded instantaneously? In this case, you can replace the discrete growth equation with a continuous growth equation:

$$Y_t = Y_0 e^{rt}$$

Using this formulation after 2 years you have:

$$Y_2 = 100e^{0.03(2)} = 106.18$$

and after five years you have:

$$Y_5 = 100e^{0.03(5)} = \$116.18$$

Solving for Time

Using the continuous time formula, you can also easily solve the following types of problems: In how many years will you have \$200 in the bank?

$$\begin{aligned} 200 &= 100e^{0.03t} \\ \ln(2) &= 0.03t \\ t &= 23.105 \end{aligned}$$

Doubling Time

An interesting feature of these exponential growth/decay equations is that the amount of time it takes to go from \$100 to \$200 is the same amount of time it takes to go from \$200 to \$400.

What matters is the amount of growth, not the starting point.

To get a better sense of the relationship between Y_0 and Y_t, let's solve the problem one more time going from \$200 to \$400.

$$\begin{aligned} Y_t &= Y_0 e^{rt} \\ 400 &= 200 e^{0.03t} \\ \ln(2) &= 0.03t \\ t &= 23.105 \end{aligned}$$

Effect of Interest Rate

The other interesting feature of the growth equation is how r affects the doubling (or half) time.
Intuitively, the higher the interest rate, the sooner you should double your money. Just to see that this is, indeed, the case, let's solve the problem with the interest rate set to 6%. How long until we go from \$100 to \$200?

$$\begin{aligned} 200 &= 100e^{0.06t} \\ \ln(2) &= 0.06t \\ t &= 11.552 \end{aligned}$$

So we doubled the interest rate and we cut the doubling time by half (from 23 years to 11.5 years).