## Calculus Review Session

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## Topics to be covered

1. Functions and Continuity
2. Solving Systems of Equations
3. Derivatives (one variable)
4. Exponentials and Logarithms
5. Derivatives (multiple variables)
6. Integration
7. Optimization
***Please ask questions at any time!

## Functions and Continuity

- Function: $f(x)$
- Mathematical relationship between variables (input "x", output "y")
- Each input ( $x$ ) is related to one and only one output (y).
- Easy graphical test: does an arbitrary vertical line intersect in more than one place?


## Functions, continuous functions

- Are these functions?

$$
y=|x|
$$



$$
x^{2}+y^{2}=1
$$



$|y|=x$


$$
\begin{aligned}
& y=\begin{array}{cc}
\begin{array}{cc}
1 & \text { for } 0 \leq x<1 \\
2 & \text { for } 1 \leq x<3 \\
3 & \text { for } x \geq 3
\end{array}
\end{array} \\
& \underbrace{}_{\mathrm{x}} \begin{array}{l}
\text { Yes! } \bullet \\
\bullet
\end{array}
\end{aligned}
$$



- Continuous: a function for which small changes in $x$ result in small changes in $y$.
- No holes, skips, or jumps
- Intuitive test: can you draw the function without lifting your pen from the paper?


## Functions, continuous functions

- Are these continuous functions?


$$
x^{2}+y^{2}=1
$$

No! (not à function)


No! $\begin{aligned} & |y|=x \\ & \text { not a }\end{aligned}$ function)


$$
\begin{aligned}
& y=\left\{\begin{array}{cc}
1 & \text { for } 0 \leq x<1 \\
2 & \text { for } 1 \leq x<3 \\
3 & \text { for } x \geq 3
\end{array}\right. \\
& \text { No! } \\
& \begin{array}{cc}
\bullet & \bullet \\
0 & 0
\end{array} \\
& \hline
\end{aligned}
$$



## Solving Systems of Equations

## Solving systems of linear equations

- Count number of equations, number of unknowns
- If \# equations = \# unknowns, unique solution might be possible Example:
A. $y=x$
B. $y=2-x$

Answer: $x=1, y=1$

- If \# equations < \# unknowns, no unique solution. Example:
- $a=1-2 b$. What is the value of $b$ ? Impossible. Any value works
- If \# equations > \# unknowns, generally no solution that satisfies all of them. Example:
- A and B above and third equation: $y=1+x$
- Algebraic solutions
- Graphical solution


## Economics Example: Algebraic Approach

- Economics Example

1. Supply: $Q^{\text {supplied }}=2 *$ Price
2. Demand: $Q^{\text {demanded }}=4-2 *$ Price
3. "Market Clearing": $Q^{\text {supplied }}=Q^{\text {demanded }}$

- Substitute equations (1) and (2) into (3) and solve for Price.
- $2 *$ Price $=4-2 *$ Price
$-4 *$ Price $=4 \rightarrow$ Price $=1$
- Substitute price back into (1) and (2)
- $Q^{\text {supplied }}=2 *$ Price $=2$
- $Q^{\text {demanded }}=4-2 *$ Price $=2$
- Price $=1, Q^{\text {supplied }}=2, Q^{\text {demanded }}=2$


## Economics Example: Graphical Approach

- Economics Example

1. Supply: $Q^{\text {supplied }}=2 *$ Price
2. Demand: $Q^{\text {demanded }}=4-2 *$ Price
3. "Market Clearing": $Q^{\text {supplied }}=Q^{\text {demanded }}$

- Rearrange equations so they're all in the same " $y=$ " terms
- (1) becomes Price $=\frac{1}{2} Q^{\text {supplied }}$
- (2) becomes Price $=\frac{1}{-2}\left(Q^{\text {demanded }}-4\right)=2-\frac{1}{2} Q^{\text {demanded }}$



## Derivatives

## Differentiation, also known as taking the derivative (one variable)

- The derivative of a function is the rate of change of the function.
- Often denote it as:

$$
f^{\prime}(x) \text { or } \frac{d}{d x} f(x) \text { or } \frac{d f}{d x}
$$

- Be comfortable using these interchangeably
- We can interpret the derivative (at a particular point) as the slope of the tangent line at that point.

- If there is a small change in $x$, how much does $f(x)$ change?


## Differentiation, also known as taking the derivative

 (one variable)- Linear function: derivative is a constant, the slope

$$
\text { (if } y=m x+b \text {, then } \frac{d y}{d x}=m \text { ) }
$$

- Nonlinear function: derivative is not constant, but rather a function of x .

Linear Function

$$
f(x)=2 x-2
$$



Non-Linear Function
$f(x)=-x^{2}+5 x-2$


## Differentiable functions

- Differentiable: A function is differentiable at a point when there's a defined derivative at that point.
- Algebraic test: if you know the equation and can solve for the derivative
- Graphical test: "slope of tangent line of points from the left is approaching the same value as slope of the tangent line of the points from the right"
- Intuitive graphical test: "As I zoom in, does the function tend to become a straight line?"
- Continuously differentiable function: differentiable everywhere x is defined.
- That is, everywhere in the domain. If not stated, then negative to positive infinity.
- If differentiable, then continuous.
- However, a function can be continuous and not differentiable! (e.g., $y=|x|$ )
- Why do we care?
- When a function is differentiable, we can use all the power of calculus.


## Functions, continuous functions

- Are these continuously differentiable functions?


$$
x^{2}+y^{2}=1
$$

No! (not à function)

No! $\begin{gathered}|y|=x \\ \text { not a }\end{gathered}$ function)


## Rules of differentiation (one variable)

- Power rule

$$
\begin{aligned}
& -y=k x^{a} \rightarrow \frac{d y}{d x}=a k x^{a-1} \\
& - \text { E.g., } y=2 x^{3} \rightarrow \frac{d y}{d x}=6 x^{2}
\end{aligned}
$$

- Derivative of a constant
- $y=k \rightarrow \frac{d y}{d x}=0$
- E.g., $y=3 \rightarrow \frac{d y}{d x}=0$
- Chain rule

$$
\begin{aligned}
& -y=f(g(x)) \rightarrow \frac{d y}{d x}=\frac{d f}{d g} \frac{d g}{d x} \\
& \\
& - \text { E.g., } y=(1+7 x)^{2} \rightarrow \frac{d y}{d x}=2(1+7 x) * 7=14+98 x \\
& \text { 17 }-f(g(x))=g(x)^{2} \quad g(x)=1+7 x
\end{aligned}
$$

## Rules of differentiation (one variable)

- Addition rule
- $y=f(x)+g(x) \rightarrow \frac{d y}{d x}=\frac{d f}{d x}+\frac{d g}{d x}$
- E.g., $y=2 x+x^{3} \rightarrow \frac{d y}{d x}=2+3 x^{2}$
- Product rule

$$
\begin{aligned}
& -y=f(x) * g(x) \rightarrow \frac{d y}{d x}=\frac{d f}{d x} g(x)+\frac{d g}{d x} f(x) \\
& \text { - E.g., } y=x^{2}(3 x+1) \rightarrow \frac{d y}{d x}=2 x(3 x+1)+3 x^{2}=9 x^{2}+2 x
\end{aligned}
$$

- Quotient rule
$-y=\frac{f(x)}{g(x)} \rightarrow \frac{d y}{d x}=\frac{\frac{d f}{d x} g(x)-\frac{d g}{d x} f(x)}{g(x)^{2}}$
- E.g., $y=\frac{x^{2}}{(3 x+1)} \rightarrow \frac{d y}{d x}=\frac{2 x(3 x+1)-3 x^{2}}{(3 x+1)^{2}}$


## Second, third and higher derivatives

- Derivative of the derivative
- Easy: Differentiate the function again (and again, and again...)
- Some functions (polynomials without fractional or negative exponents) reduce to zero, eventually
$-y=x^{2}$
- First derivative $=2 x$
- Second derivative $=2$
- Third derivative $=0$
- Other functions may not reduce to zero: e.g., $f(x)=e^{x}$


## Exponentials and Logarithms

## Exponents and logarithms (logs)

- Logs are incredibly useful for understanding exponential growth and decay
- half-life of radioactive materials in the environment
- growth of a population in ecology
- effect of discount rates on investment in energy-efficient lighting
- Logs are the inverse of exponentials, just like addition:subtraction and multiplication:division

$$
y=b^{x} \leftrightarrow \log _{b}(y)=x
$$

- In practice, we most often use base $e$ (Euler's number, 2.71828182846...).
- We write this as "In": $\ln x=\log _{e} x$.
- Sometimes, we also use base 10.
- When in doubt, use natural log
- Important! In Excel, LOG() is base 10 and LN() is natural log


## Rules of logarithms

- Logarithm of exponential function: $\ln \left(e^{x}\right)=x$
- $\log$ of exponential function (more generally): $\ln e^{g(x)}=g(x)$
- Exponential of log function: $e^{\ln x}=x$
- More generally, $e^{\ln h(x)}=h(x)$
- Log of products: $\ln (x y)=\ln x+\ln y$
- Log of ratio or quotient: $\ln \left(\frac{x}{y}\right)=\ln x-\ln y$
- Log of a power: $\ln \left(x^{k}\right)=k \ln x$
- E.g., $\ln \left(x^{2}\right)=2 \ln (x)$


## Derivatives of logarithms

- $\frac{d}{d x} \ln x=\frac{1}{x}$. This is just a rule. You have to memorize it.
- What about $\frac{d}{d x} \ln 2 x$ ?
- Chain Rule: $\frac{d}{d x} \ln (2 x)=\frac{1}{2 x} 2=\frac{1}{x}$
- Or, use the fact that $\ln 2 x=\ln 2+\ln x$ and take the derivative of each term. (Simpler.)
- Also, this means $\frac{d}{d x} \ln k x=\frac{d}{d x}(\ln k+\ln x)=\frac{1}{x}$
- ... for any constant $k>0$.
- ( $\ln A$ is defined only for $A>0$.)
- In general for $\frac{d}{d x} \ln g(x)$, where $\mathrm{g}(\mathrm{x})$ is any function of x , use the Chain Rule.
- Why useful? Log-changes give percentages: $d \ln (x)=\frac{d x}{x}$


## Derivatives of exponents

- $\frac{d}{d x} e^{x}=e^{x}$. This is just a rule. You have to memorize it.
- What about $\frac{d}{d x} e^{2 x}$ ?
- To solve, rewrite so that $f(g(x))=e^{g(x)}$ and $g(x)=2 x$.
- The Chain Rule tells us that $\frac{d}{d x} f(g(x))=\frac{d f}{d g} \frac{d g}{d x}$.
- $\frac{d f}{d g}=e^{g(x)}$ and $\frac{d g}{d x}=2$
- $\frac{d}{d x} f(g(x))=\frac{d f}{d g} \frac{d g}{d x}=e^{g(x)} * 2=2 e^{2 x}$
- Analogous to how $\frac{d}{d x} \ln (x)$ is the growth as a percentage, $e^{x}$ grows at a rate proportional to its current value
- E.g., if a population level is given by $y=100 e^{0.05 t}$, where $t$ is time in years, then:
${ }^{24}-\frac{d y}{d t}=0.05 *\left(100 e^{0.05 t}\right)=0.05 y$, so it is growing at a rate of $5 \%$.


## Why Exponentials and Logs?

- Numerous applications
- Interest rates (for borrowing or investment)
- Decomposition of radioactive materials
- Growth of a population in ecology
- Annual vs. continuous compounding
- See examples in pdf notes
- With exponential growth/decay, doubling time or half-life is constant, depending on $r$.
- As $r$ increases, doubling time or half-life is shorter.
- Intuitive: faster rate of increase or decay.


## Derivatives with functions of more than one variable

## Partial and total derivatives

- All the previous stuff about derivatives was based on $y=f(x)$ : one input variable and one output.
- What about multivariate relationships?
- E.g., $f(x, y, z)=x^{2} y^{3}-2 x z$
- Demand for energy-efficient appliances depends on income and prices
- Growth of a prey population depends on natural reproduction rate, rate of growth of predator population, environmental carrying capacity for prey
- Forest size depends on trees planted, trees harvested, natural growth rates, etc.
- Partial derivatives let us express change in the output variable given a small change in the input variable, with other variables still in the mix


## Guidelines for partial derivatives

- Partial derivatives denoted $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$, etc. ("curly d")
- Differentiate each term one by one, holding other variables constant
- Suppose you're differentiating with respect to x.
- If a term has $x$ in it, take the derivative with respect to $x$.
- If a term does not have $x$ in it, it's a constant with respect to $x$.
- The derivative of a constant with respect to $x$ is zero.
- Examples

$$
\begin{array}{lll}
-f=x^{2} y^{3}+2 x+y & \rightarrow & \frac{\partial f}{\partial x}=2 x y^{3}+2
\end{array} \quad \frac{\partial f}{\partial y}=x^{2} * 3 y^{2}+1 .
$$

## Guidelines for partial derivatives

- Cross-partial derivatives: for $f(x, y)$, first $\frac{\partial}{\partial x}$, then $\frac{\partial}{\partial y}$
- Denoted $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ or $\frac{\partial^{2} f}{\partial y \partial x}$
- "How does the $\frac{\partial f}{\partial x}$ slope change as y changes?"
- Or the other way around. They're equivalent.
- That is, you could take $\frac{\partial}{\partial y}$ first and then take $\frac{\partial}{\partial x}$ of the result:
$-f=x^{2} y^{3}+2 x \rightarrow \frac{\partial f}{\partial x}=2 x y^{3}+2 \quad \frac{\partial f}{\partial y}=x^{2} * 3 y^{2}$
$-\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=2 x * 3 y^{2} \Leftrightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=2 x * 3 y^{2}$


## Total derivatives / differentials

- Total Derivative: $f(x, y)$. What if both x and y are changing together?
- Represent the change in a multivariate function with respect to all variables
- Economics Example: $f(x, y)=$ profits, where $x=$ price, $y=$ \#customers
- Partial Deriv.: how much would profits change if we $\uparrow$ price, holding \#customers constant?
- Total Deriv.: how much would profits change if we $\uparrow$ price, accounting for resulting loss in customers?
- More general example: both $x$ and $y$ are changing over time, so doesn't make sense to hold one constant
- Sum of the partial derivatives for each variable, multiplied by the change in that variable
- $d f=\frac{\partial f}{\partial x} d x+\frac{\partial f}{\partial y} d y$
- See https://en.wikipedia.org/wiki/Total derivative


## Integration

## Integration

- Integral of a function is "the area under the curve" (or the line)
- Integration (aka "antiderivative") is the inverse of differentiation
- Just like addition is inverse of subtraction
- Just like exponents are inverse of logarithms
- Thus: the integral of the derivative is the original function plus a constant of integration. Or,
$\int\left(\frac{d f}{d x}\right) d x=f(x)+c$



## Integration

- Integration is useful for recovering total functions when we start with a function representing a change in something
- Location, when starting with speed
- Value of natural capital stock (e.g., forest) when we have a function representing its growth rate
- Total demand when we start with marginal demand
- Numerous applications in statistics, global climate change, etc.
- Two functions that have the same derivative can vary by a constant (thus, the constant of integration)
- Example: $\frac{d}{d x}\left(x^{2}+4000\right)=2 x$
- Also, $\frac{d}{d x}\left(x^{2}-30\right)=2 x$
- So we write $\int 2 x d x=x^{2}+c$, where c is any constant.


## Rules of integration (one variable)

- Rules can be thought of as "reversing" the rules for derivatives
- Power rule
- $\int x^{a} d x=\frac{1}{a+1} x^{a+1}+c$
- E.g., $\int x^{2} d x=\frac{1}{3} x^{3}+c$
- Integral of a constant
$-\int k d x=k x+c$
- Exponential Rule
$-\int e^{x} d x=e^{x}+c$
- Logarithmic Rule
$-\int \frac{1}{x} d x=\ln (x)+c$
- Integral of sums
$-\int(f(x)+g(x)) d x=\int f(x) d x$

$$
+\int g(x) d x
$$

- Can pull constants out

$$
-\int k f(x) d x=k \int f(x) d x
$$

## Indefinite vs. Definite integrals

- With indefinite integral, we recover the function that represents the reverse of differentiation
$-\int \frac{d f}{d x} d x=f(x)+c$
- E.g., $\int x^{2} d x=\frac{1}{3} x^{3}+c$
- This function would give us the area under the curve
- ... but as a function, not a number
- With definite integral, we solve for the area under the curve between two points
$-\int_{a}^{b} \frac{d f}{d x} d x=f(b)-f(a)$
- And so we should get a number
- ... or a function, in a multivariate context (but we won't talk about that today)
- "How far did we move between 1 pm and 2 pm?"


## Definite integrals

- Basic approach
- Compute the indefinite integral
- Drop the constant of integration
- Evaluate the integral at the upper limit of integration
- Evaluate the integral at the lower limit of integration
- Calculate the difference: (upper - lower)
- Example:
- $\int_{2}^{6}\left(3 x^{2}+2\right) d x$
- Indefinite integral: $x^{3}+2 x \quad$ (without constant of integration)
- Evaluate this at the upper limit: $6^{3}+2 * 6$
- Evaluate this at the lower limit: $2^{3}+2 * 2$
- Take the difference: $\left(6^{3}+2 * 6\right)-\left(2^{3}+2 * 2\right)=228-12=\mathbf{2 1 6}$
- In math notation: $\int_{2}^{6}\left(3 x^{2}+2\right) d x=\left.\left(x^{3}+2 x\right)\right|_{2} ^{6}=\mathbf{2 1 6}$

Optimization

## Optimization: Finding minimums and maximums

- Use first derivatives to see how afunction is changing

$$
\begin{aligned}
& \frac{d y}{d x}>0 \text { : function is increasing } \\
& \frac{d y}{d x}<0 \text { : function is decreasing }
\end{aligned}
$$

- What is happening when $\frac{d y}{d x}=0$ ?
- One possibility: function is "turning around"
" This is a "critical point"
- Another possibility: inflection point. (consider $y=x^{3}$ at $x=0$.)





## Procedure for finding minimums and maximums

- Take first derivative
- Where does first derivative equal zero?
- These are candidate points for min or max ("critical points")
- Example: $f(x)=-x^{2}+5 x-2$
$-\frac{d f}{d x}=-2 x+5:=0 \Rightarrow x=\frac{5}{2}=2.5$
- Take second derivative

- Use second derivatives to determine how the change is changing
- Minimum: $\frac{d y}{d x}=0$ and $\frac{d^{2} y}{d x^{2}}>0$
- Maximum: $\frac{d y}{d x}=0$ and $\frac{d^{2} y}{d x^{2}}<0$
- See technical notes on next slide.
- Ex: $\frac{d^{2} f}{d^{2} x}=-2<0 \Rightarrow$ Maxiumum



## Finding minimums and maximums (technical notes)

- Technically, $\frac{d y}{d x}=0$ is a necessary condition for a min or max. (In order to a point to be a min or max, $\frac{d y}{d x}$ must be zero.)
- $\frac{d^{2} y}{d x^{2}}>0$ is a sufficient condition for a minimum, and $\frac{d^{2} y}{d x^{2}}<0$ is a sufficient condition for a maximum.
- But, these are not necessary conditions.
- That is, there could be a minimum at a point where $\frac{d^{2} y}{d x^{2}}=0$.
- This is a technical detail that you almost certainly don't need to know until you take higher-level applied math.
- For a good, quick review of necessary and sufficient conditions, watch this 3-minute video: https://www.khanacademy.org/partner-content/wi-phi/critical-thinking/v/necessary-sufficient-conditions


## Inflection points

- Inflection point is where the function changes from concave to convex, or vice versa
- Second derivative tells us about concavity of the original function
- Inflection point: $\frac{d y}{d x}=0$ and $\frac{d^{2} y}{d x^{2}}=0$
- Technical: $\frac{d^{2} y}{d x^{2}}=0$ is a necessary but not sufficient condition for inflection point
- That's enough for our purposes.
- Just know that an inflection point is where $\frac{d y}{d x}=0$ but the point is not a min or a max.
- For more information I recommend:
- https://www.mathsisfun.com/calculus/maxima-minima.html (easiest)
- http://clas.sa.ucsb.edu/staff/lee/Max\ and\ Min's.htm
- http://clas.sa.ucsb.edu/staff/lee/Inflection\ Points.htm
- http://www.sosmath.com/calculus/diff/der13/der13.html
- http://mathworld.wolfram.com/InflectionPoint.html (most technical)


## Further resources

- These slides, notes, sample problems (see email)
- Strang textbook: http://ocw.mit.edu/resources/res-18-001-calculus-online-textbook-spring-2005/textbook/
- Strang videos at http://ocw.mit.edu/resources/res-18-005-highlights-of-calculus-spring-2010/ (see "highlights of calculus")
- Khan Academy videos: https://www.khanacademy.org/math
- Math(s) Is Fun: https://www.mathsisfun.com/links/index.html (10 upwards; algebra, calculus)
- Numerous other resources online. Find what works for you.
- Wolfram Alpha computational knowledge engine at http://www.wolframalpha.com/
- Often useful for checking intuition or calculations
- Excellent way to get a quick graph of a function

