Calculus Review Session

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Schedule

Time	Event
2:00-2:20	Introduction
2:20-2:40	Functions; systems of equations
2:40-3:00	Derivatives, also known as differentiation
3:00-3:20	Exponents and logarithms
3:20-3:30	Break
3:30-3:45	Partial and total derivatives
3:45-4:05	Integration
4:05-4:15	Exponential growth and decay
4:15-4:30	Optimization

***Please ask questions at any time!

Introduction

- Groups of three
- Please take 2 to 2 ½ minutes per person to share:
  - Your name
  - Where you're from
  - Something that's not on your resume
  - Something you're passionate about related to the environment
  - A pleasant memory involving mathematics

Topics to be covered

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Solving systems of linear equations

- Count number of variables, number of unknowns
- Algebraic solutions
  - Solve by substitution
  - Solve by elimination
- Graphical solution
- Examples

Functions, continuous functions

- Function:
  - Mathematical relationship of two variables (input, output)
  - Each input (x) is related to one and only one output (y).
  - Easy graphical test: does an arbitrary vertical line intersect in more than one place?
- Continuous: a function for which small changes in x result in small changes in y.
  - Intuitive test: can you draw the function without lifting your pen from the paper?
  - No holes or skips
  - Vertical asymptotes “in the middle” of the function

Differentiable functions

- Differentiable: A function is differentiable at a point when there's a defined derivative at that point.
  - Algebraic test (if you know the equation and can solve for the derivative)
  - Series or limit test
    - See Strang text
    - Or http://www.mathsisfun.com/calculus/differentiable.html and http://www.mathsisfun.com/calculus/continuity.html
  - Graphical test: "slope of tangent line of points from the left is approaching the same value as slope of the tangent line of the points from the right"
  - Intuitive graphical test: "As I zoom in, does the function tend to become a straight line?"
- Continuously differentiable: differentiable everywhere x is defined.
  - That is, everywhere in the domain. If not stated, then negative to positive infinity.
- If differentiable, then continuous.
  - However, a function can be continuous and not differentiable!
- Why do we care?
  - When a function is differentiable, we can use all the power of calculus.

Differentiation, also known as taking the derivative (one variable)

- The derivative of a function is the rate of change of the function.
- We can interpret the derivative (at a particular point) as the slope of the tangent line (at that point).
- If there is a small change in x, how much does y change?
- Linear function: derivative is a constant, the slope

$$\text{(if } y = mx + b\text{, then } \frac{dy}{dx} = m\text{)}$$

- Nonlinear function: derivative is not constant, but rather a function of x.

Rules of differentiation (one variable)

- Power rule
- Derivative of a constant
- Chain rule
- Product rule
- Quotient rule
- Addition rule (not on handout)

Combining differentiation with simultaneous equations

- Identify where two functions have the same slope
- Identify a point of tangency

- Method:
  - Take derivatives of both functions
  - Set derivatives equal to each other
  - Solve the system of equations

Second, third and higher derivatives

- Differentiate the function again
- (and again, and again…)

- Some functions (polynomials without fractional or negative exponents) reduce to zero, eventually
- Other functions may not reduce to zero: e.g., $f(x) = e^x$

Exponents and logarithms (logs)

- Logs are incredibly useful for understanding exponential growth and decay
  - half-life of radioactive materials in the environment
  - growth of a population in ecology
  - effect of discount rates on investment in energy-efficient lighting
- Logs are the inverse of exponentials, just like addition:subtraction and multiplication:division

$$y = b^x \leftrightarrow \log_b(y) = x$$

- In practice, we most often use base e (Euler's number, 2.71828182846…). We write this as ln: $\ln x = \log_e x$.
- Sometimes, we also use base 10.

Rules of logarithms

- Logarithm of exponential function: $\ln(e^x) = x$
  - log of exponential function (more generally): $\ln e^{g(x)} = g(x)$

- Exponential of log function: $e^{\ln x} = x$
  - More generally, $e^{\ln h(x)} = h(x)$

- Log of products: $\ln(xy) = \ln x + \ln y$
- Log of ratio or quotient: $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
- Log of a power: $\ln(x^2) = 2\ln x$

Derivatives of logarithms

- $\frac{d}{dx}\ln x = \frac{1}{x}$. This is just a rule. You have to memorize it.
- What about $\frac{d}{dx}\ln 2x$?
  - Chain Rule + use the fact that $\frac{d}{dx}\ln x = \frac{1}{x}$
  - Or, use the fact that $\ln 2x = \ln 2 + \ln x$ and take the derivative of each term. (Simpler.)
  - Also, this means $\frac{d}{dx}\ln kx = \frac{d}{dx}(\ln k + \ln x) = \frac{1}{x}$
    - … for any constant $k > 0$.
    - ($\ln A$ is defined only for $A > 0$.)
- In general for $\frac{d}{dx}\ln g(x)$, where g(x) is any function of x, use the Chain Rule.

Derivatives of exponents

- $\frac{d}{dx}e^x = e^x$. This is just a rule. You have to memorize it.
- What about $\frac{d}{dx}e^{2x}$?
  - Chain Rule + the rule about $\frac{d}{dx}e^x = e^x$.
  - To solve, rewrite so that $f(g) = e^g$ and $g(x) = 2x$.
  - The Chain Rule tells us that $\frac{d}{dx}f\left(g(x)\right) = \frac{df}{dg}\frac{dg}{dx}$.
    - $\frac{df}{dg} = e^g$
    - $\frac{dg}{dx} = 2$
    - $\frac{d}{dx}f\left(g(x)\right) = \frac{df}{dg}\frac{dg}{dx} = e^g * 2$
    - $\frac{d}{dx}f\left(g(x)\right) = \frac{df}{dg}\frac{dg}{dx} = 2e^{2x}$

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Partial and total derivatives

- All the previous stuff about derivatives was based on $y = f(x)$: one input variable and one output.
- What about multivariate relationships?
  - Demand for energy-efficient appliances depends on income and price
  - Growth of a prey population depends on natural reproduction rate, rate of growth of predator population, environmental carrying capacity for prey
- Partial derivatives let us express change in the output variable given a small change in the input variable, with other variables still in the mix

Guidelines for partial derivatives

- Use the addition rule for derivatives to distinguish each term
- Differentiate each term one by one
- Suppose you're differentiating with respect to x.
  - If a term has x in it, take the derivative with respect to x.
  - If a term does not have x in it, it's a constant with respect to x.
  - The derivative of a constant with respect to x is zero.
- Examples (on the board)
- Cross-partial derivatives: for $f(x, y)$, first $\frac{d}{dx}$, then $\frac{d}{dy}$
  - Or the other way around. They're equivalent.
  - That is, you could take $\frac{d}{dy}$ first and then take $\frac{d}{dx}$ of the result.
- More complicated for more complicated terms, e.g. Chain Rule & Product Rule & …

Total derivatives

- Represent the change in a multivariate function with respect to all variables
- Sum of the partial derivatives for each variable, multiplied by the change in that variable
- See the handout (pdf) for the formula and an example

Integration

- Integral of a function is "the area under the curve" (or the line)
- Integration is the reverse of differentiation
  - Just like addition is reverse of subtraction
  - Just like exponents are reverse of logarithms
  - Thus: the integral of the derivative is the original function plus a constant of integration. Or,

$$\int \left(\frac{df}{dx}\right) dx = f(x) + c$$

Integration

- Integration is useful for recovering total functions when we start with a function representing a change in something
  - Value of natural capital stock (e.g., forest) when we have a function representing a flow
  - Total demand when we start with marginal demand
  - Numerous applications in statistics, global climate change, etc.
- Two functions that have the same derivative can vary by a constant (thus, the constant of integration)
  - Example: $\frac{d}{dx}(x^2 + 4000) = 2x$
  - Also, $\frac{d}{dx}(x^2 - 30) = 2x$
  - So we write $\int 2x\,dx = x^2 + c$, where c is any constant.

Rules of integration

- Power rule
- Exponential rule
- Logarithmic rule
- Integrals of sums
- Integrals involving multiplication
- Integration with substitution, integration by parts
  - Not covered today
  - See Strang text (sections 5.4, 7.1)
  - Probably some videos online as well – e.g., Khan Academy

Definite integrals

- With indefinite integral, we recover the function that represents the reverse of differentiation
  - This function would give us the area under the curve
  - … but as a function, not a number
- With definite integral, we solve for the area under the curve between two points
  - And so we should get a number
  - … or a function, in a multivariate context (but we won't talk about that today)
- Basic approach
  - Compute the indefinite integral
  - Drop the constant of integration
  - Evaluate the integral at the upper limit of integration (call this value_upper)
  - Evaluate the integral at the lower limit of integration (call this value_lower)
  - Calculate (value_upper – value_lower)

Exponential growth and decay

- Numerous applications
  - Interest rates (for borrowing or investment)
  - Decomposition of radioactive materials
  - Growth of a population in ecology
- Annual compounding (or decay)
- Continuous compounding (or decay)
- See examples in pdf notes
- Note that doubling time or half-life is constant, depending on r.
- As r increases, doubling time or half-life is shorter.
  - Intuitive: faster rate of increase or decay.

Optimization: Finding minimums and maximums

- Applications throughout economics and science
- Use first derivatives to characterize how the function is changing

$$\frac{dy}{dx} > 0\text{: function is increasing}$$

$$\frac{dy}{dx} < 0\text{: function is decreasing}$$

- What is happening when $\frac{dy}{dx} = 0$?
  - One possibility: function is no longer increasing/decreasing
  - This is a "critical point"
  - Another possibility: inflection point. (consider $y = x^3$ at $x = 0$.)

Procedure for finding minimums and maximums

- Take first derivative
  - Where does first derivative equal zero?
  - These are candidate points for min or max (“critical points”)
- Take second derivative
  - Use second derivatives to characterize how the change is changing
  - Minimum: $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$
  - Maximum: $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$
  - See technical notes on next slide.

Finding minimums and maximums (technical notes)

- Technically, $\frac{dy}{dx} = 0$ is a *necessary* condition for a min or max. (In order to a point to be a min or max, $\frac{dy}{dx}$ must be zero.)
- $\frac{d^2y}{dx^2} > 0$ is a *sufficient* condition for a minimum, and $\frac{d^2y}{dx^2} < 0$ is a *sufficient* condition for a maximum.
  - But, these are not *necessary* conditions.
  - That is, there could be a minimum at a point where $\frac{d^2y}{dx^2} = 0$.
  - This is a technical detail that you almost certainly don't need to know until you take higher-level applied math.
  - For a good, quick review of *necessary* and *sufficient* conditions, watch this 3-minute video: https://www.khanacademy.org/partner-content/wi-phi/critical-thinking/v/necessary-sufficient-conditions

Inflection points

- Inflection point is where the function changes from concave to convex, or vice versa
- Second derivative tells us about concavity of the original function
- Inflection point: $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$
  - Technical: $\frac{d^2y}{dx^2} = 0$ is a necessary but not sufficient condition for inflection point
- That's enough for our purposes.
  - Just know that an inflection point is where $\frac{dy}{dx} = 0$ but the point is not a min or a max.
  - For more information I recommend:
    - https://www.mathsisfun.com/calculus/maxima-minima.html (easiest)
    - http://clas.sa.ucsb.edu/staff/lee/Max%20and%20Min's.htm
    - http://clas.sa.ucsb.edu/staff/lee/Inflection%20Points.htm
    - http://www.sosmath.com/calculus/diff/der13/der13.html
    - http://mathworld.wolfram.com/InflectionPoint.html (most technical)

Further resources

- These slides, notes, sample problems (see email)
- Strang textbook: http://ocw.mit.edu/resources/res-18-001-calculus-online-textbook-spring-2005/textbook/
- Strang videos at http://ocw.mit.edu/resources/res-18-005-highlights-of-calculus-spring-2010/ (see "highlights of calculus")
- Khan Academy videos: https://www.khanacademy.org/math
- Math(s) Is Fun: https://www.mathsisfun.com/links/index.html (10 upwards; algebra, calculus)
- Numerous other resources online. Find what works for you.
- Wolfram Alpha computational knowledge engine at http://www.wolframalpha.com/
  - Often useful for checking intuition or calculations
  - Excellent way to get a quick graph of a function