Math Review – practice questions
Some sourced from http://tutorial.math.lamar.edu, others original
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August 2016

1. Simultaneous equations. Solve for x and y:

a. $3x - y = 7$
$2x + 3y = 1$

Solve by substitution:

$$3x - y = 7 \rightarrow y = 3x - 7$$

$$2x + 3(3x - 7) = 1$$
$$2x + 9x - 21 = 1$$
$$11x = 22$$
$$x = 2$$
$$y = 3x - 7 \rightarrow y = -1$$

b. $x - 2y + 3z = 7$
$2x + y + z = 4$
$-3x + 2y - 2z = -10$

There are a few ways to go about this, but I often find it easiest to start with elimination and then move to substitution.
Let's call those equations (1), (2) and (3), in order from top to bottom.
Start with solving by elimination. Use the equations (1) and (3):

$$x - 2y + 3z = 7$$
$$-3x + 2y - 2z = -10$$

Sum these equations to get a new one. Let's call this equation (4a):

$$-2x + z = -3$$

From equation (4a) we also know that

$$z = -3 + 2x$$

Let's call this (4b).

Now, substitute (4b) into (2) to find that

$$2x + y + (-3 + 2x) = 4$$
$$4x + y = 7$$

And substitute (4b) into (1) to find that

$$x - 2y + 3(-3 + 2x) = 7$$
$$7x - 2y - 9 = 7$$
$$7x - 2y = 16$$

Now, we know that both $4x + y = 7$ and $7x - 2y = 16$. Let's go back to elimination, and multiply the first of these new equations by 2 (on both sides):

$$8x + 2y = 14$$
$$7x - 2y = 16$$

Summing these, we get $15x = 30$, so $x = 2$.
Now that we have this piece of the puzzle, we can go back to our earlier two-variable equations:

$4x + y = 7$ means that if $x = 2$ then $y = -1$.

And, $z = -3 + 2x$ means that if $x = 2$ then $z = 1$.

So the solution is $x = 2$, $y = -1$, $z = 1$.

If you're ever uncertain about the solution, it's useful to go back and check that the combination works in the original equations given. If one of them doesn't work out, then you know the solution you found is not correct.

c. $2x + 5y = -1$
$-10x - 25y = 5$

Solve by elimination:

$$5(2x + 5y) = 5(-1) \rightarrow 10x + 25y = -5$$

$$\begin{cases} 10x + 25y = -5 \\ -10x - 25y = 5 \end{cases}$$

The sum of the two equations shows $0 = 0$. Thus, the equations represent the same line. There is no unique solution of (x,y).

2. True/false. The following are functions. Why or why not?

a. $x^2 + y^2 = 4$

False.
The algebraic approach is to solve for y:

$$y^2 = 4 - x^2$$
$$y = \pm\sqrt{4 - x^2}$$

The $\pm$ sign provides a hint that for some values of x, there can be more than one value of y. We can verify this by showing that if $x = 1$ then $y = \pm\sqrt{3}$. Therefore, this is not a function because there is at least one value of x for which there is more than one value of y.

The graphical approach is to graph the function and show that it is a circle (centered on the origin with a radius of 2). We can then draw a vertical line (say, at $x = 1$) and show that y can take on two distinct values for this single value of x.

b. $f(x) = \begin{cases} -x, x < 0 \\ x, x \geq 0 \end{cases}$

True.
The algebraic approach is to show that there is no value of x for which $f(x)$ takes on more than one value (you can try any value of x and see that no value will produce more than one value for $f(x)$).
The graphical approach is to graph the function and see that we cannot draw a vertical line such that $f(x)$ takes on more than one value.

3. Differentiate each of the following functions:

a. $f(x) = (6x^3 - x)(10 - 20x)$

Use the product rule and the power rule:

$u = (6x^3 - x)$	$v = (10 - 20x)$
$u' = (18x^2 - 1)$	$v' = (-20)$

$$\frac{d}{dx}u(x)v(x) = vu' + uv'$$

$$vu' = (10 - 20x)(18x^2 - 1)$$
$$uv' = (6x^3 - x)(-20)$$

$$\frac{d}{dx}f(x) = (10 - 20x)(18x^2 - 1) + (6x^3 - x)(-20)$$

You can simplify this to:

$$\frac{d}{dx}f(x) = -480x^3 + 180x^2 + 40x - 10$$

b. $g(x) = 3e^x + e^{3x}$
Use the addition rule, the chain rule, the power rule and the exponent rule.

$$g'(x) = \frac{d}{dx}3e^x + \frac{d}{dx}e^{3x}$$

$$g'(x) = 3e^x + 3e^{3x}$$

c. $g(x) = (6x^3 - x - 4)^2$
Use the chain rule, power rule and addition rule.

$$g(x) = (f(x))^2$$
$$f(x) = 6x^3 - x - 4$$

$$g'(x) = \left(\frac{dg}{df}\right)\left(\frac{df}{dx}\right)$$

$$= (2f(x))(18x^2 - 1)$$

$$= (12x^3 - 2x - 8)(18x^2 - 1)$$

d. $W(z) = \frac{3z^2+9}{2-z}$

Use the quotient rule and power rule. Quotient rule: $\frac{vu' - uv'}{v^2}$

$u = 3z^2 + 9$	$v = 2 - z$
$u' = 6z$	$v' = -1$

$$W'(z) = \frac{(2-z)(6z) - (3z^2+9)(-1)}{(2-z)^2}$$

$$W'(z) = \frac{12z - 6z^2 + 3z^2 + 9}{(2-z)^2}$$

$$W'(z) = \frac{-3z^2 + 12z + 9}{(2-z)^2}$$

e. $f(x) = e^x + \ln x$

Use the addition rule and log/exponent rules.

$$f'(x) = e^x + \frac{1}{x}$$

f. $f(x) = e^x + 10x^3 \ln x$

Use the addition rule, product rule, and log/exponent rules.

For the second term:

$u = 10x^3$	$v = \ln x$
$u' = 30x^2$	$v' = \frac{1}{x} = x^{-1}$

$$f'(x) = e^x + (10x^3x^{-1}) + 30x^2 \ln x$$

$$f'(x) = e^x + x^2(10 + 30 \ln x)$$

g. $y(x) = \sqrt[3]{x^2}(2x - x^2)$

Use the product rule, power rule, and addition rule.

$u = x^{2/3}$	$v = (2x - x^2)$
$u' = {}^2/_3\, x^{-1/3}$	$v' = (2 - 2x)$

$$y'(x) = uv' + vu'$$
$$y'(x) = (x^{2/3})(2-2x) + (2x - x^2)({}^2/_3\, x^{-1/3})$$

Depending on the context, this may be sufficient, or you may need to simplify further.

4. Find all first and second order partial derivatives (including cross-partials) for:

a. $f(x,y) = x^4 + 6\sqrt{y} - 10$

Start by observing that $\sqrt{y} = y^{\frac{1}{2}}$. Then:

$\frac{\partial f}{\partial x} = 3x^3$	$\frac{\partial f}{\partial y} = 3y^{-\frac{1}{2}}$	$\frac{\partial^2 f}{\partial x \partial y} = 0$
$\frac{\partial^2 f}{\partial x^2} = 9x^2$	$\frac{\partial^2 f}{\partial y^2} = -\frac{3}{2}y^{-\frac{3}{2}}$	

b. $f(x,y) = 3x^2 + 2xy - 4x^{-1}y^2$

$\frac{\partial f}{\partial x} = 6x + 2y + 4x^{-2}y^2$	$\frac{\partial f}{\partial y} = 2x - 8x^{-1}y$	$\frac{\partial^2 f}{\partial x \partial y} = 2 + 8x^{-2}y$
$\frac{\partial^2 f}{\partial x^2} = 6 - 8x^{-3}y^2$	$\frac{\partial^2 f}{\partial y^2} = -8x^{-1}$	

In both cases, note that we can calculate $\frac{\partial^2 f}{\partial x \partial y}$ as either $\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]$ or $\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]$, and find the same result.

5. Find the total derivatives of the following functions:

a. $f(x,y) = x^2 + xy + y^2$

The total derivative is defined as $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$. So, first let's calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$\frac{\partial f}{\partial x} = 2x + y$ and $\frac{\partial f}{\partial y} = x + 2y$.

Thus, the total derivative is $(2x + y)dx + (x + 2y)dy$.

b. $f(x,y) = \ln x + yz + x^2y^2z^2$

In this case, the total derivative is defined as $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$.

$$\frac{\partial f}{\partial x} = \frac{1}{x} + 2xy^2z^2 \qquad \frac{\partial f}{\partial y} = z + 2x^2yz^2 \qquad \frac{\partial f}{\partial z} = y + 2x^2y^2z$$

Thus, the total derivative is $\left(\frac{1}{x} + 2xy^2z^2\right)dx + (z + 2x^2yz^2)dy + (y + 2x^2y^2z)dz$.

6. Identify all minima, maxima, and inflection points of the following functions:

a. $f(x) = 2x^2 + 6x + 7$

A necessary condition for any minimum or maximum is that $\frac{df}{dx} = 0$. A necessary condition for any inflection point is that $\frac{d^2f}{dx^2} = 0$.[1] We'll start looking for minima and maxima by taking the first derivative, then finding the value or values of x that would make that first derivative equal to zero:

$$f'(x) = 4x + 6$$

The equation $4x + 6 = 0$ is true for only one value of x: $x = -\frac{3}{2}$.
To identify if this is a minimum, a maximum or an inflection point, we look at the second derivative:

$$f''(x) = 4$$

Since $f''(x) > 0$ always (including when $x = -\frac{3}{2}$), we conclude that $x = -\frac{3}{2}$ is a minimum.

If you forget the second derivative rule, in a pinch, there are two other ways to verify this. First, we could plot the function – pick a few points and calculate the corresponding values.

Or, we could plug in a few values in the neighborhood of $x = -\frac{3}{2}$ and see if the function has a higher or lower value than it does at $x = -\frac{3}{2}$. For instance, we could evaluate the function:
At $x = -\frac{3}{2}: f(x) = \frac{5}{2}$
And at $x = -1: f(x) = 3$
And at $x = -2: f(x) = 3$
Since the function has a smaller value at $x = -\frac{3}{2}$, we conclude this is a minimum.

Since $f''(x)$ is never equal to zero, there are no inflection points for this function.

b. $g(x) = -x^2 - 4x + 4$

Similar to (a), we start by taking the first derivative and identifying when it equals zero.

$$g'(x) = -2x - 4$$

This equals zero at only one point: $x = -2$. To identify if this is a minimum, maximum, or inflection point, we look at the second derivative:

$$g''(x) = -2$$

[1] This clarifies a point from the last few minutes of the calculus review. For a little more insight (and some graphs showing some inflection points) see http://www.mathsisfun.com/calculus/inflection-points.html. For more technical detail than you need (but a clarification of the second-derivative necessary condition), see http://mathworld.wolfram.com/InflectionPoint.html.

Since $g''(x) < 0$ always (including when $x = -2$), we conclude that $x = -2$ is a maximum. (Again, we could plot the function and verify, or plug in when $x = -2$ and at some neighboring values.) Since $g''(x)$ is never equal to zero, there are no inflection points for this function.

7. Find the x and y intercepts of $y = 3x - 6$. (The intercepts are where the function intersects the x and y axes.) What are the (x,y) coordinates at these intercepts?

When the function intersects the y axis, $x = 0$. If $x = 0$ then $y = 3x - 6 = -6$. That is, the coordinates (x,y) when the function intersects the y axis are (0,-6).

When the function intersects the x axis, $y = 0$. If $y = 0$ then $3x - 6 = 0$. The value of x for which this is true is $x = 2$. Thus, the coordinates (x,y) when the function intersects the x axis are (2,0).

8. Find all values of x such that these two functions have the same slope. Do the two functions touch at this point? How do you know?

$$f(x) = 3x - 1$$
$$g(x) = x^2 - 4$$

If the two functions have the same slope, that means that $f'(x) = g'(x)$. We want to find the value(s) of x for which this is true. We'll start by calculating $f'(x)$ and $g'(x)$:

$$f'(x) = 3$$
$$g'(x) = 2x$$

If $f'(x) = g'(x)$ then $3 = 2x$. This is true for only one value of x: $x = \frac{3}{2}$.

At $x = \frac{3}{2}$, $f(x) = 3\left(\frac{3}{2}\right) - 1 = \frac{7}{2}$. $\qquad g(x) = \left(\frac{3}{2}\right)^2 - 4 = -\frac{7}{4}$.

These are not the same value, so when the two functions have the same slope (i.e., at $x = \frac{3}{2}$), they do not touch.

9. Evaluate the following integrals if possible:

a.

$$\int x^4 + 3x - 9\,dx$$

Use the addition rule and power rule. Note that this is an indefinite integral, so we will include the constant c in the answer.

$$\frac{x^5}{5} + \frac{3x^2}{2} - 9x + c$$

b.

$$\int_{-1}^{2} y^2 + 2y\,dy$$

Use the addition rule and power rule. Note that this is a definite integral.

$$\left[\frac{y^3}{3} + y^2\right]_{-1}^{2} = \left[\left(\frac{2^3}{3} + 2^2\right) - \left(\frac{(-1)^3}{3} + (-1)^2\right)\right]$$

$$= \left[\left(\frac{8}{3} + 4\right) - \left(-\frac{1}{3} + 1\right)\right]$$

$$= \left(\frac{20}{3}\right) - \left(\frac{2}{3}\right)$$

$$= \frac{18}{3} = 6$$

c.

$$\int_{-3}^{1} 6x^2 - 5x + 2\,dx$$

Use the addition rule and power rule. Again, this is a definite integral.

$$\left[2x^3 - \frac{5}{2}x^2 + 2x\right]_{-3}^{1} = \left[\left(2(1)^3 - \frac{5}{2}(1)^2 + 2(1)\right) - \left(2(-3)^3 - \frac{5}{2}(-3)^2 + 2(-3)\right)\right]$$

$$= \left(2 - \frac{5}{2} + 2\right) - \left(-54 - \frac{45}{2} - 6\right)$$

$$= \left(\frac{3}{2}\right) - \left(-\frac{165}{2}\right)$$

$$= -\frac{168}{2} = 84$$

10. The growth of a colony of bacteria is given by the equation:

$$Q = Q_0 e^{0.195t}$$

If there are initially 500 bacteria present ($Q_0 = 500$) and t is given in hours, determine each of the following.

a. How many bacteria are there after half a day?

Half a day is 12 hours. So at this point, $Q = 500e^{0.195*12}$. (Without a calculator, this is the best we would expect you to do.)

With a calculator, you could calculate that $0.195 * 12 = 2.34$ and $e^{2.34} = 10.38$. So $Q = 500(10.38) = 5190.6$ (approximately).

b. How long will it take before there are 10,000 bacteria in the colony?

The idea here is to set Q=10,000 and solve for t. That is:

$$10000 = 500e^{0.195t}$$

$$20 = e^{0.195t}$$

$$\ln 20 = \ln e^{0.195t}$$

$$\ln 20 = 0.195t$$

Without a calculator, this is the best we would expect you to do.
With a calculator, you could go on to calculate:

$$2.996 = 0.195t$$

$$t = 15.36 \text{ hours}$$

c. For a different species of bacteria, the growth of a colony is given by the equation $Q = Q_0 e^{0.5t}$. If there are initially 500 bacteria present ($Q_0 = 500$) and t is given in hours, would you expect this species to reach 10000 bacteria in the colony in more time, less time, or about the same amount of time as the species in which $Q = Q_0 e^{0.195t}$?

Less time. Intuitively, since the growth rate r is greater in magnitude, this species grows faster than the one addressed in part (a) and (b). Thus, we would expect this species to grow to 10000 in *less time* than the species addressed in (a) and (b).

You can also verify this mathematically using the same approach as in (b) (and show that the colony would grow to 10000 in about 6 hours).

11. After Ella finishes her MEM at Duke, she receives a job offer with a $15,000 signing bonus. She puts the entire bonus into an account that earns interest at a rate of 6 percent (annually) for 5 years. Assuming she does not make any further deposits or withdrawals, determine how much money will be in the account at the end of 5 years, if interest is:

a. Compounded annually

The general formula for calculating the balance with discrete compounding is $Y_t = Y_0(1+r)^t$.

In this case $Y_0 = 15000$, $r = 0.06$, and $t = 5$ (years). Thus, $Y_t = 15000(1.06)^5 =$ \$20073.

Note: Without a calculator, we'd just expect you to answer $15000(1.06)^5$.

b. Compounded monthly

We use the same formula for discrete compounding, but if interest is compounded monthly then we need to denote r and t in units of months rather than years.
Then $r = \frac{0.06}{12} = 0.005$ and $t = 5 * 12 = 60$.
Then, $Y_t = 15000(1.005)^{60} = \$20233$. This makes sense – it's a little larger than the answer in (a), since the interest is compounded a little more frequently.

Note: Without a calculator, we'd just expect you to answer $15000(1.005)^{60}$.

c. Compounded continuously

This time, we use the formula for continuous compounding: $Y_t = Y_0 e^{rt}$. We can keep r and t in units of years.

$$Y_t = 15000e^{0.06\times 5} = 15000e^{0.3} = \$20248$$

It makes sense that this is slightly higher than in (b) because the interest is compounded more frequently. And again, without a calculator, we'd just expect you to answer $15000e^{0.3}$.

Suggestions for additional exercises:

- The final slide in the PowerPoint presentation provides several resources for additional explanations and problems.
- Another good resource (that provided the source for some of these problems) is http://tutorial.math.lamar.edu/.
- The Wolfram Alpha search engine is a great resource for plotting functions and calculating algebraic and numerical answers. You can make up your own problems and check your answers with this search engine. For instance, the answer to 9b (with a graph!) is here: https://www.wolframalpha.com/input/?i=integral+of+y%5E2%2B2y+dy+from+-1+to+2.