Calculus Review Session

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Topics to be covered

1. Functions and Continuity
2. Solving Systems of Equations
3. Derivatives (one variable)
4. Exponentials and Logarithms
5. Derivatives (multiple variables)
6. Integration
7. Optimization

***Please ask questions at any time!

Functions and Continuity

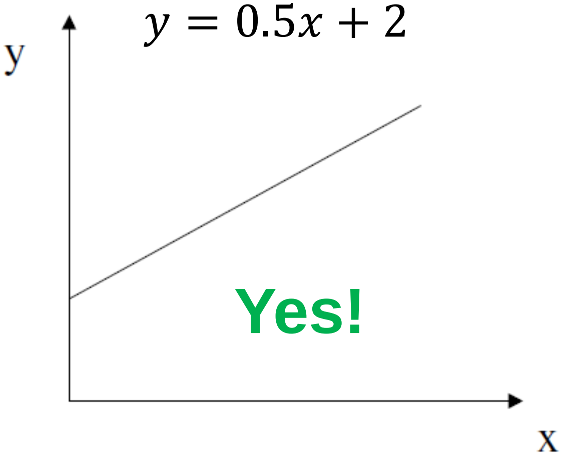
Functions, continuous functions

- Function: $f(x)$
  - Mathematical relationship between variables (input “x”, output “y”)
  - Each input (x) is related to one and only one output (y).
  - Easy graphical test: does an arbitrary vertical line intersect in more than one place?

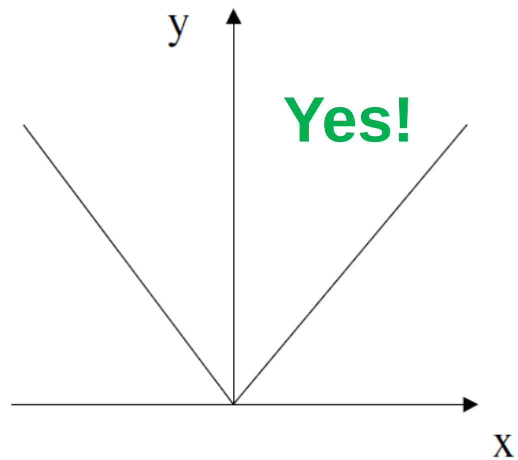
Functions, continuous functions

- Are these functions?

$y = 0.5x + 2$



$y = |x|$



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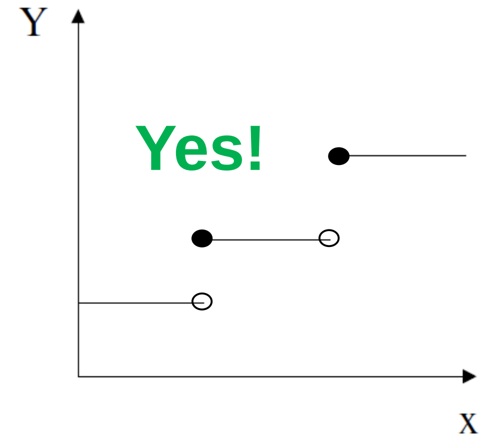
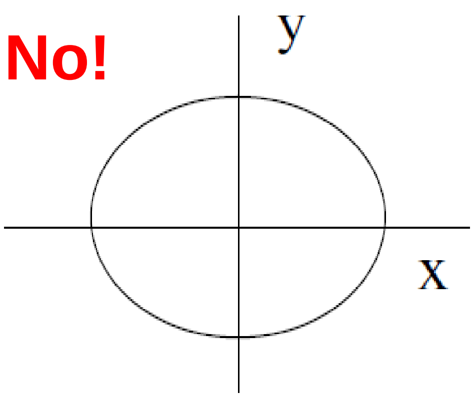
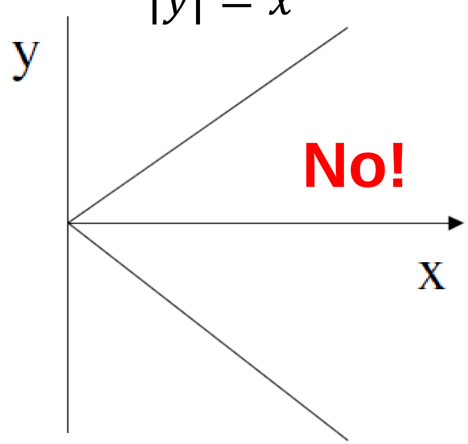


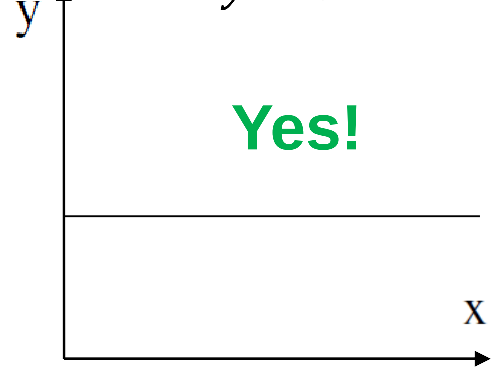
$x^2 + y^2 = 1$



$|y| = x$



$y = 3$



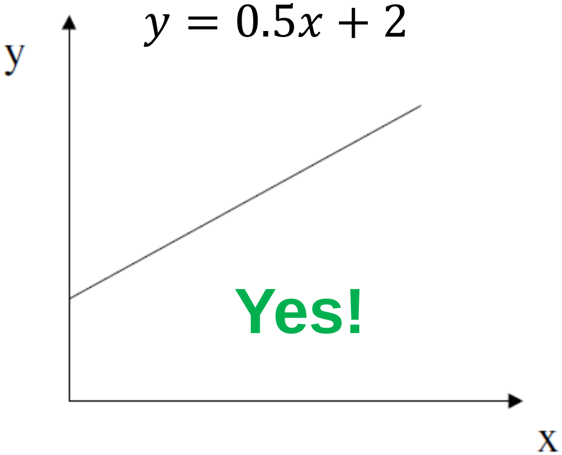
Functions, continuous functions

- **Continuous**: a function for which small changes in x result in small changes in y.
  - No holes, skips, or jumps
  - Intuitive test: can you draw the function without lifting your pen from the paper?

Functions, continuous functions

- Are these ***continuous*** functions?

$y = 0.5x + 2$



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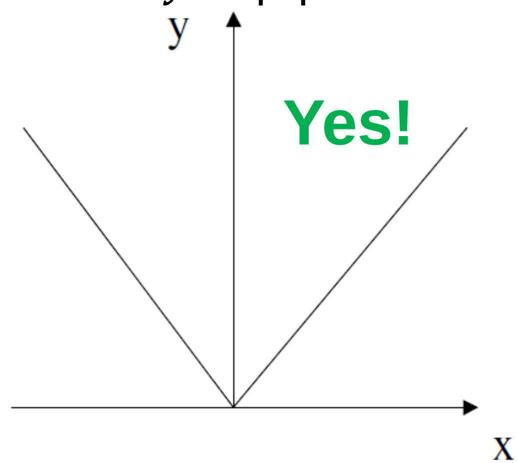

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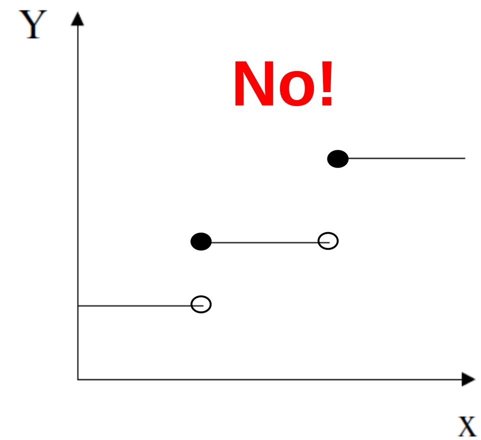

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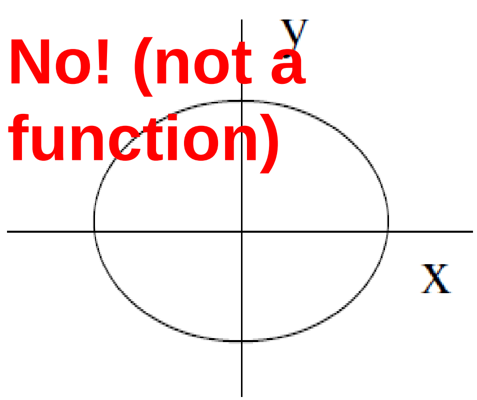

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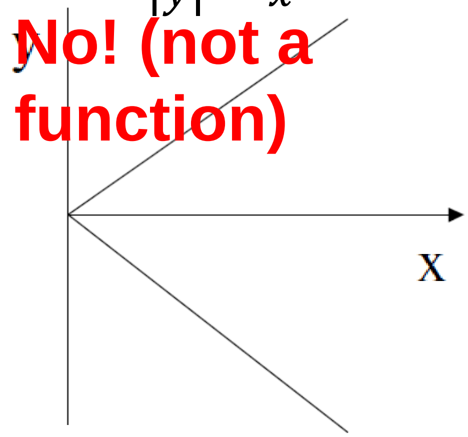

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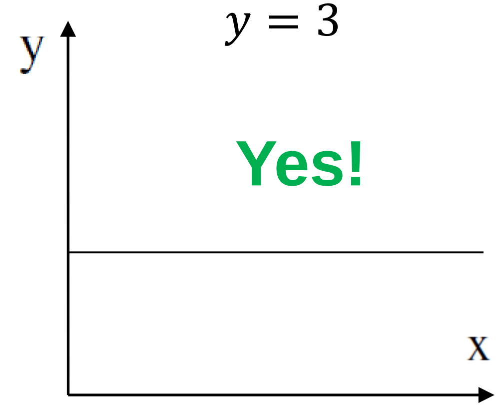


Solving Systems of Equations

Solving systems of linear equations

- Count number of equations, number of unknowns
  - If # equations = # unknowns, unique solution might be possible Example:
    - *A.* $y = x$
    - *B.* $y = 2 - x$

    Answer: $x = 1,\ y = 1$
  - If # equations < # unknowns, no unique solution. Example:
    - $\mathrm{a} = 1 - 2b$. What is the value of b? Impossible. Any value works
  - If # equations > # unknowns, generally no solution that satisfies all of them. Example:
    - A and B above **<u>and third equation:</u>** $y = 1 + x$
- Algebraic solutions
- Graphical solution

Economics Example: Algebraic Approach

- Economics Example
  1. Supply: $Q^{supplied} = 2 * Price$
  2. Demand: $Q^{demanded} = 4 - 2 * Price$
  3. "Market Clearing": $Q^{supplied} = Q^{demanded}$

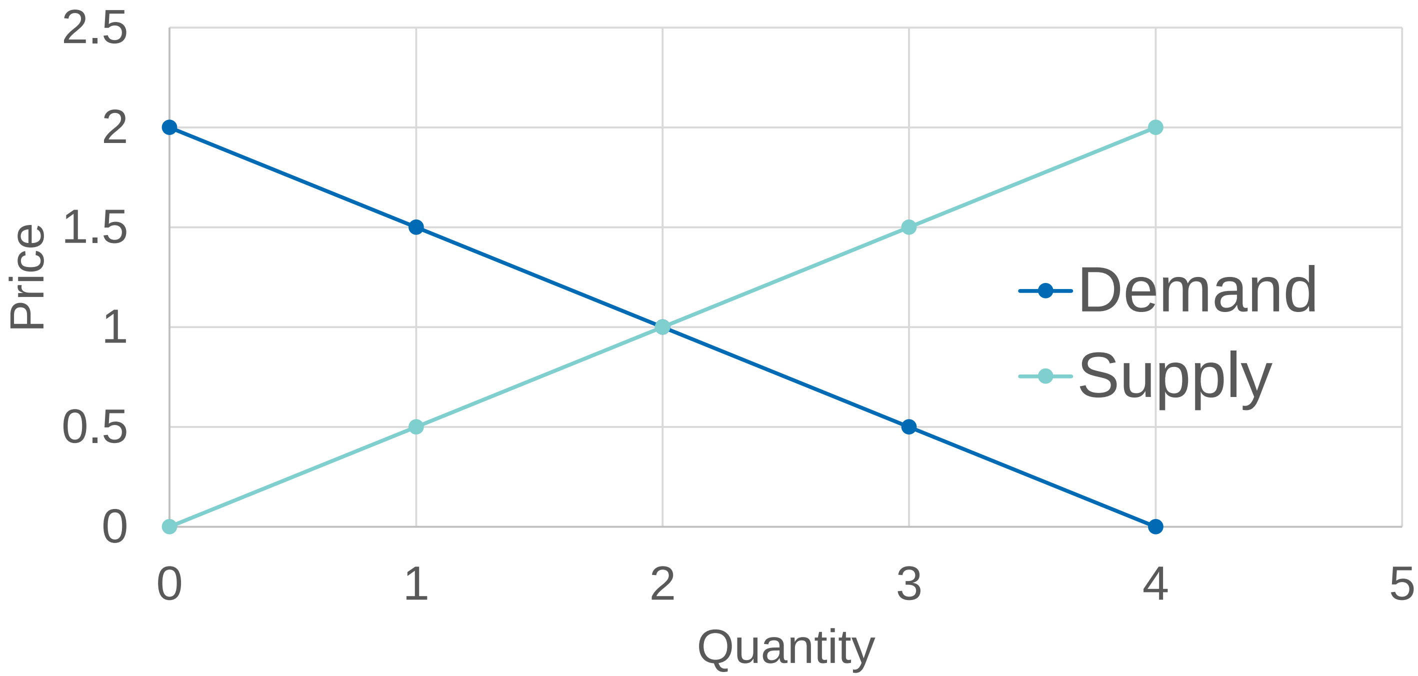
- Substitute equations (1) and (2) into (3) and solve for Price.
  - $2 * Price = 4 - 2 * Price$
  - $4 * Price = 4 \rightarrow Price = 1$

- Substitute price back into (1) and (2)
  - $Q^{supplied} = 2 * Price = 2$
  - $Q^{demanded} = 4 - 2 * Price = 2$

- $\boldsymbol{Price = 1,\ Q^{supplied} = 2,\ Q^{demanded} = 2}$

Economics Example: Graphical Approach

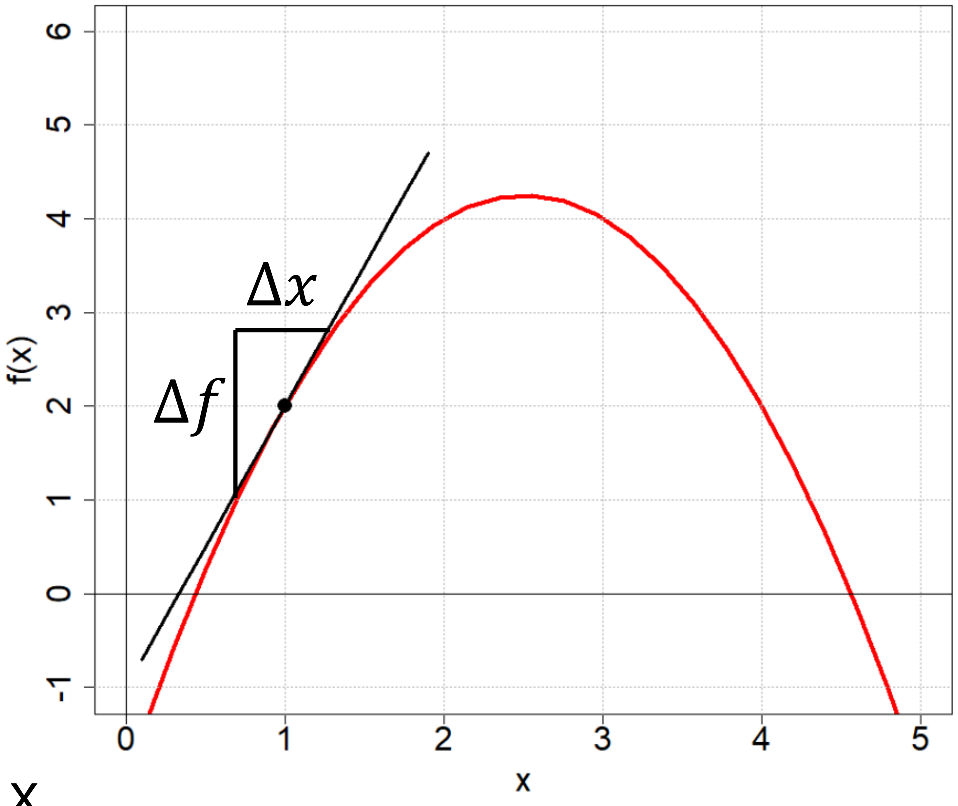
- Economics Example
  1. Supply: $Q^{supplied} = 2 * Price$
  2. Demand: $Q^{demanded} = 4 - 2 * Price$
  3. “Market Clearing”: $Q^{supplied} = Q^{demanded}$
- Rearrange equations so they’re all in the same “y=” terms
  - (1) becomes $Price = \frac{1}{2} Q^{supplied}$
  - (2) becomes $Price = \frac{1}{-2}(Q^{demanded} - 4) = 2 - \frac{1}{2} Q^{demanded}$



Derivatives

Differentiation, also known as taking the derivative (one variable)

- The **derivative** of a function is the rate of change of the function.
  - Often denote it as:
    $f'(x)$ or $\frac{d}{dx}f(x)$ or $\frac{df}{dx}$
  - Be comfortable using these interchangeably



- We can interpret the derivative (at a particular point) as the slope of the tangent line at that point.
  - If there is a small change in x, how much does f(x) change?

Differentiation, also known as taking the derivative (one variable)

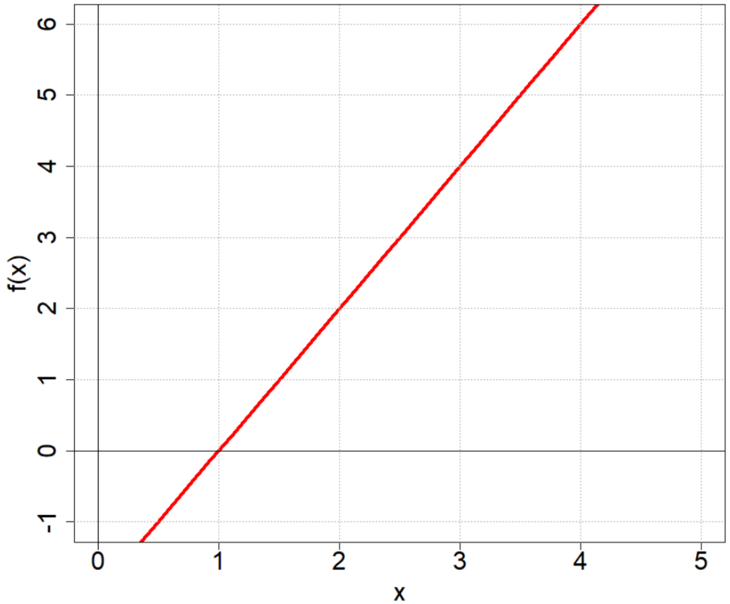
- Linear function: derivative is a constant, the slope

(if $y = mx + b$, then $\frac{dy}{dx} = m$)

- Nonlinear function: derivative is not constant, but rather a function of x.

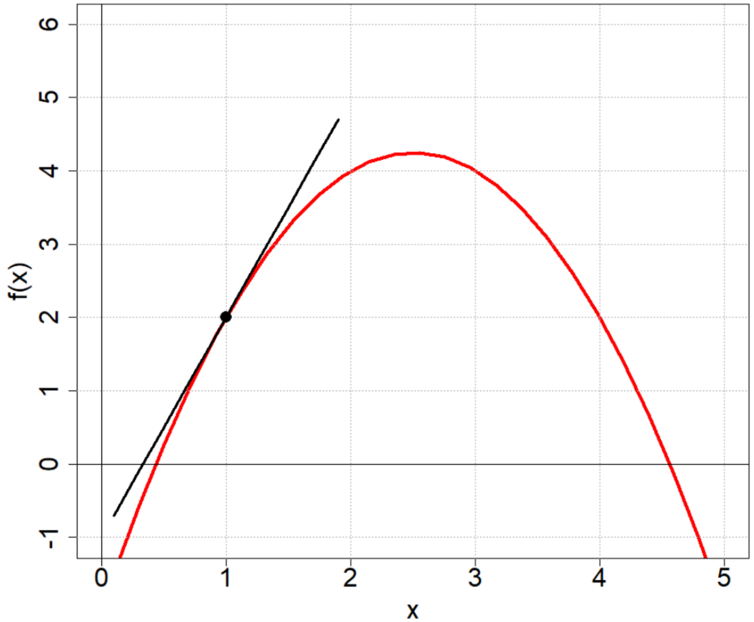
Linear Function

$f(x) = 2x - 2$



Non-Linear Function

$f(x) = -x^2 + 5x - 2$



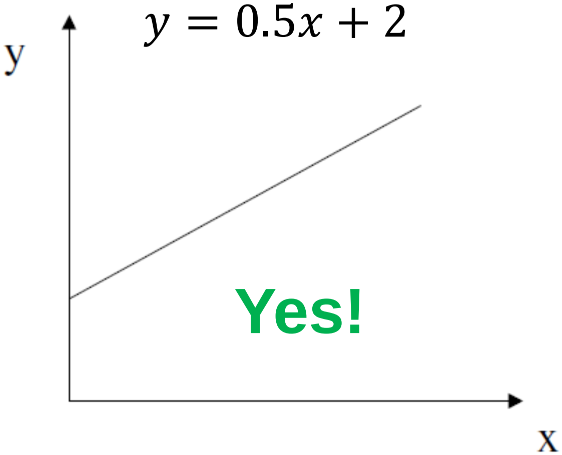
Differentiable functions

- Differentiable: A **function is differentiable at a point** when there's a defined derivative at that point.
  - Algebraic test: if you know the equation and can solve for the derivative
  - Graphical test: “slope of tangent line of points from the left is approaching the same value as slope of the tangent line of the points from the right”
  - Intuitive graphical test: “As I zoom in, does the function tend to become a straight line?”
- **Continuously differentiable function**: differentiable everywhere x is defined.
  - That is, everywhere in the domain. If not stated, then negative to positive infinity.
- If differentiable, then continuous.
  - However, a function can be continuous and not differentiable! (e.g., $y = |x|$)
- Why do we care?
  - When a function is differentiable, we can use all the power of calculus.

Functions, continuous functions

- Are these ***continuously differentiable*** functions?

$$y = 0.5x + 2$$



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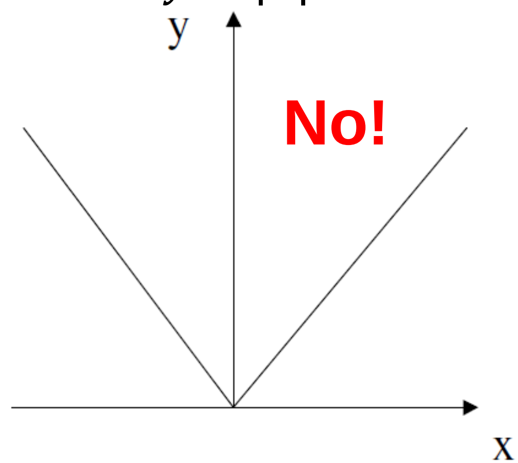

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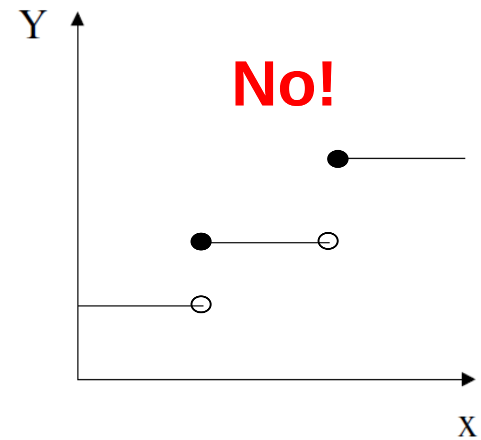
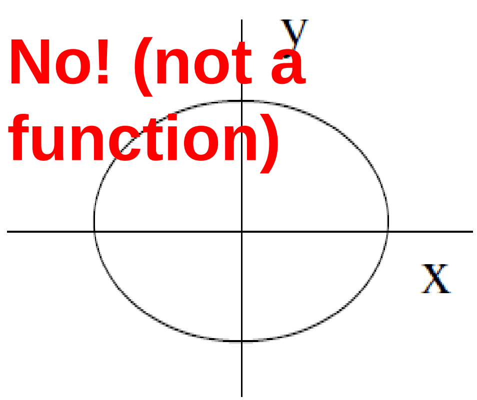
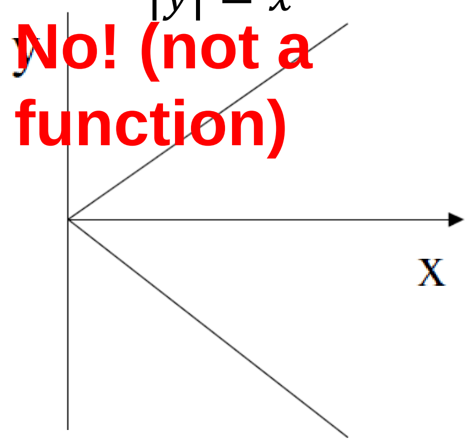


$$x^2 + y^2 = 1$$



$$|y| = x$$



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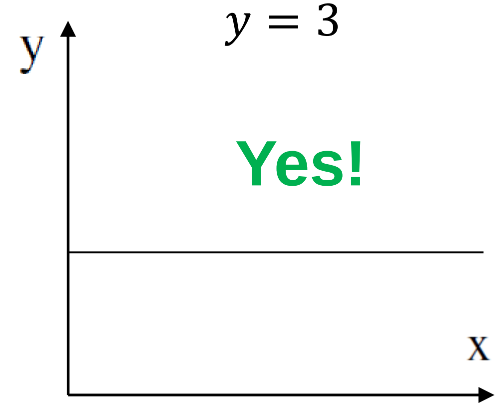


Rules of differentiation (one variable)

- Power rule
  - $y = kx^a \rightarrow \frac{dy}{dx} = akx^{a-1}$
  - E.g., $y = 2x^3 \rightarrow \frac{dy}{dx} = 6x^2$
- Derivative of a constant
  - $y = k \rightarrow \frac{dy}{dx} = 0$
  - E.g., $y = 3 \rightarrow \frac{dy}{dx} = 0$
- Chain rule
  - $y = f\big(g(x)\big) \rightarrow \frac{dy}{dx} = \frac{df}{dg}\frac{dg}{dx}$
  - E.g., $y = (1 + 7x)^2 \rightarrow \frac{dy}{dx} = 2(1 + 7x) * 7 = 14 + 98x$
  - $f(g(x)) = g(x)^2$ $\quad g(x) = 1 + 7x$

Rules of differentiation (one variable)

- Addition rule
  - $y = f(x) + g(x) \rightarrow \frac{dy}{dx} = \frac{df}{dx} + \frac{dg}{dx}$
  - E.g., $y = 2x + x^3 \rightarrow \frac{dy}{dx} = 2 + 3x^2$
- Product rule
  - $y = f(x) * g(x) \rightarrow \frac{dy}{dx} = \frac{df}{dx} g(x) + \frac{dg}{dx} f(x)$
  - E.g., $y = x^2(3x + 1) \rightarrow \frac{dy}{dx} = 2x(3x + 1) + 3x^2 = 9x^2 + 2x$
- Quotient rule
  - $y = \frac{f(x)}{g(x)} \rightarrow \frac{dy}{dx} = \frac{\frac{df}{dx} g(x) - \frac{dg}{dx} f(x)}{g(x)^2}$
  - E.g., $y = \frac{x^2}{(3x+1)} \rightarrow \frac{dy}{dx} = \frac{2x(3x+1) - 3x^2}{(3x+1)^2}$

Second, third and higher derivatives

- Derivative of the derivative
- Easy: Differentiate the function again (and again, and again…)

- Some functions (polynomials without fractional or negative exponents) reduce to zero, eventually
  - $y = x^2$
  - First derivative = $2x$
  - Second derivative = 2
  - Third derivative = 0

- Other functions may not reduce to zero: e.g., $f(x) = e^x$

Exponentials and Logarithms

Exponents and logarithms (logs)

- Logs are incredibly useful for understanding exponential growth and decay
  - half-life of radioactive materials in the environment
  - growth of a population in ecology
  - effect of discount rates on investment in energy-efficient lighting
- Logs are the inverse of exponentials, just like addition:subtraction and multiplication:division

$$y = b^x \leftrightarrow \log_b(y) = x$$

- In practice, we most often use base e (Euler's number, 2.71828182846…).
  - We write this as “ln”: $\ln x = \log_e x$.
- Sometimes, we also use base 10.
- When in doubt, use natural log
  - Important! In Excel, LOG() is base 10 and LN() is natural log

Rules of logarithms

- Logarithm of exponential function: $\ln(e^x) = x$
  - log of exponential function (more generally): $\ln e^{g(x)} = g(x)$

- Exponential of log function: $e^{\ln x} = x$
  - More generally, $e^{\ln h(x)} = h(x)$

- Log of products: $\ln(xy) = \ln x + \ln y$
- Log of ratio or quotient: $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
- Log of a power: $\ln\left(x^k\right) = k \ln x$
  - E.g., $\ln(x^2) = 2\ln(x)$

Derivatives of logarithms

- $\frac{d}{dx}\ln x = \frac{1}{x}$. This is just a rule. You have to memorize it.
- What about $\frac{d}{dx}\ln 2x$?
  - Chain Rule: $\frac{d}{dx}\ln(2x) = \frac{1}{2x}2 = \frac{1}{x}$
  - Or, use the fact that $\ln 2x = \ln 2 + \ln x$ and take the derivative of each term. (Simpler.)
  - Also, this means $\frac{d}{dx}\ln kx = \frac{d}{dx}(\ln k + \ln x) = \frac{1}{x}$
    - … for any constant $k > 0$.
    - ($\ln A$ is defined only for $A > 0$.)
- In general for $\frac{d}{dx}\ln g(x)$, where g(x) is any function of x, use the Chain Rule.
- Why useful? Log-changes give percentages: $d\ln(x) = \frac{dx}{x}$

Derivatives of exponents

- $\frac{d}{dx}e^x = e^x$. This is just a rule. You have to memorize it.
- What about $\frac{d}{dx}e^{2x}$?
  - To solve, rewrite so that $f(g(x)) = e^{g(x)}$ and $g(x) = 2x$.
  - The Chain Rule tells us that $\frac{d}{dx}f\left(g(x)\right) = \frac{df}{dg}\frac{dg}{dx}$.
    - $\frac{df}{dg} = e^{g(x)}$ and $\frac{dg}{dx} = 2$
    - $\frac{d}{dx}f\left(g(x)\right) = \frac{df}{dg}\frac{dg}{dx} = e^{g(x)} * 2 = 2e^{2x}$
- Analogous to how $\frac{d}{dx}\ln(x)$ is the growth as a percentage, e^x grows at a rate proportional to its current value
  - E.g., if a population level is given by $y = 100e^{0.05t}$, where t is time in years, then:
  - $\frac{dy}{dt} = 0.05 * (100e^{0.05t}) = 0.05y$, so it is growing at a rate of 5%.

Why Exponentials and Logs?

- Numerous applications
  - Interest rates (for borrowing or investment)
  - Decomposition of radioactive materials
  - Growth of a population in ecology
- Annual vs. continuous compounding
- See examples in pdf notes
- With exponential growth/decay, doubling time or half-life is constant, depending on r.
- As r increases, doubling time or half-life is shorter.
  - Intuitive: faster rate of increase or decay.

Derivatives with functions of more than one variable

Partial and total derivatives

- All the previous stuff about derivatives was based on $y = f(x)$: one input variable and one output.
- What about multivariate relationships?
  - E.g., $f(x, y, z) = x^2y^3 - 2xz$
  - Demand for energy-efficient appliances depends on income and prices
  - Growth of a prey population depends on natural reproduction rate, rate of growth of predator population, environmental carrying capacity for prey
  - Forest size depends on trees planted, trees harvested, natural growth rates, etc.
- **Partial** derivatives let us express change in the output variable given a small change in the input variable, with other variables still in the mix

Guidelines for partial derivatives

- Partial derivatives denoted $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$, etc. ("curly d")
- Differentiate each term one by one, holding other variables constant
- Suppose you're differentiating with respect to x.
  - If a term has x in it, take the derivative with respect to x.
  - If a term does not have x in it, it's a constant with respect to x.
  - The derivative of a constant with respect to x is zero.
- Examples
  - $f = x^2y^3 + 2x + y \quad \rightarrow \quad \frac{\partial f}{\partial x} = 2xy^3 + 2 \quad \frac{\partial f}{\partial y} = x^2 * 3y^2 + 1$
  - $z = x^2y^5 + 2xy^3 \quad \rightarrow \quad \frac{\partial z}{\partial x} = 2xy^5 + 2y^3 \quad \frac{\partial z}{\partial y} = x^2 * 5y^4 + 2x * 3y^2$

Guidelines for partial derivatives

- **<u>Cross-partial derivatives</u>**: for $f(x, y)$, first $\frac{\partial}{\partial x}$, then $\frac{\partial}{\partial y}$
  - Denoted $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ or $\frac{\partial^2 f}{\partial y \partial x}$
  - "How does the $\frac{\partial f}{\partial x}$ slope change as y changes?"

- Or the other way around. They're equivalent.
  - That is, you could take $\frac{\partial}{\partial y}$ first and then take $\frac{\partial}{\partial x}$ of the result:
  - $f = x^2y^3 + 2x \quad \rightarrow \quad \frac{\partial f}{\partial x} = 2xy^3 + 2 \qquad \frac{\partial f}{\partial y} = x^2 * 3y^2$
  - $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2x * 3y^2 \quad \Leftrightarrow \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 2x * 3y^2$

Total derivatives / differentials

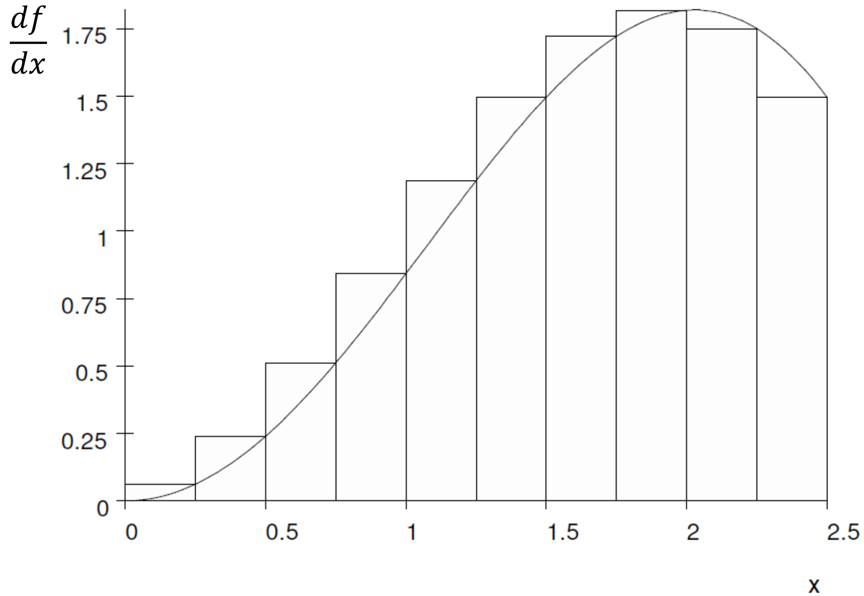
- **<u>Total Derivative</u>**: $f(x,y)$. What if both x and y are changing together?
- Represent the change in a multivariate function with respect to all variables
  - Economics Example: $f(x,y) = profits, where\ x = price,\ \ y = \#customers$
    - <u>Partial Deriv.</u>: how much would profits change if we ↑ $price$, **holding #customers constant?**
    - <u>Total Deriv.</u>: how much would profits change if we ↑ $price$, **accounting for resulting loss in customers?**
  - More general example: both x and y are changing over time, so doesn't make sense to hold one constant
- Sum of the partial derivatives for each variable, multiplied by the change in that variable
- $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$
- See https://en.wikipedia.org/wiki/Total_derivative

Integration

Integration

- Integral of a function is "the area under the curve" (or the line)
- Integration (aka “antiderivative”) is the inverse of differentiation
  - Just like addition is inverse of subtraction
  - Just like exponents are inverse of logarithms
  - Thus: the integral of the derivative is the original function plus a constant of integration. Or,

$$\int \left(\frac{df}{dx}\right) dx = f(x) + c$$



Integration

- Integration is useful for recovering total functions when we start with a function representing a change in something
  - Location, when starting with speed
  - Value of natural capital stock (e.g., forest) when we have a function representing its growth rate
  - Total demand when we start with marginal demand
  - Numerous applications in statistics, global climate change, etc.
- Two functions that have the same derivative can vary by a constant (thus, the constant of integration)
  - Example: $\frac{d}{dx}(x^2 + 4000) = 2x$
  - Also, $\frac{d}{dx}(x^2 - 30) = 2x$
  - So we write $\int 2x\, dx = x^2 + c$, where c is any constant.

Rules of integration (one variable)

- Rules can be thought of as “reversing” the rules for derivatives
- Power rule
  - $\int x^a dx = \frac{1}{a+1} x^{a+1} + c$
  - E.g., $\int x^2 dx = \frac{1}{3} x^3 + c$
- Integral of a constant
  - $\int k dx = kx + c$
- Exponential Rule
  - $\int e^x dx = e^x + c$
- Logarithmic Rule
  - $\int \frac{1}{x} dx = \ln(x) + c$
- Integral of sums
  - $\int \big(f(x) + g(x)\big) dx = \int f(x) dx + \int g(x) dx$
- Can pull constants out
  - $\int kf(x) dx = k \int f(x) dx$

Indefinite vs. Definite integrals

- With **indefinite** integral, we recover the function that represents the reverse of differentiation
  - $\int \frac{df}{dx} dx = f(x) + c$
  - E.g., $\int x^2 dx = \frac{1}{3} x^3 + c$
  - This function would give us the area under the curve
  - … but as a function, not a number

- With **definite** integral, we solve for the area under the curve between two points
  - $\int_a^b \frac{df}{dx} dx = f(b) - f(a)$
  - And so we should get a number
  - … or a function, in a multivariate context (but we won't talk about that today)
  - "How far did we move between 1pm and 2pm?"

Definite integrals

- Basic approach
  - Compute the indefinite integral
  - Drop the constant of integration
  - Evaluate the integral at the upper limit of integration
  - Evaluate the integral at the lower limit of integration
  - Calculate the difference: (upper – lower)
- Example:
  - $\int_2^6 (3x^2 + 2)dx$
  - Indefinite integral: $x^3 + 2x$ (without constant of integration)
  - Evaluate this at the upper limit: $6^3 + 2 * 6$
  - Evaluate this at the lower limit: $2^3 + 2 * 2$
  - Take the difference: $(6^3 + 2 * 6) - (2^3 + 2 * 2) = 228 - 12 = \mathbf{216}$
  - In math notation: $\int_2^6 (3x^2 + 2)dx = (x^3 + 2x)|_2^6 = \mathbf{216}$

Optimization

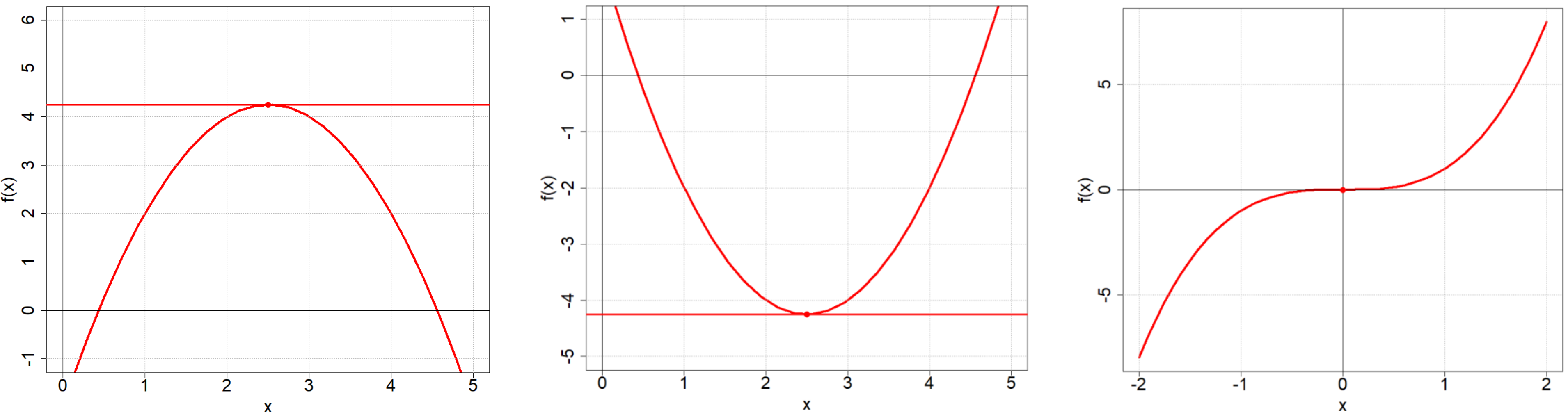
Optimization: Finding minimums and maximums

- Use first derivatives to see how afunction is changing

$$\frac{dy}{dx} > 0\text{: function is increasing}$$

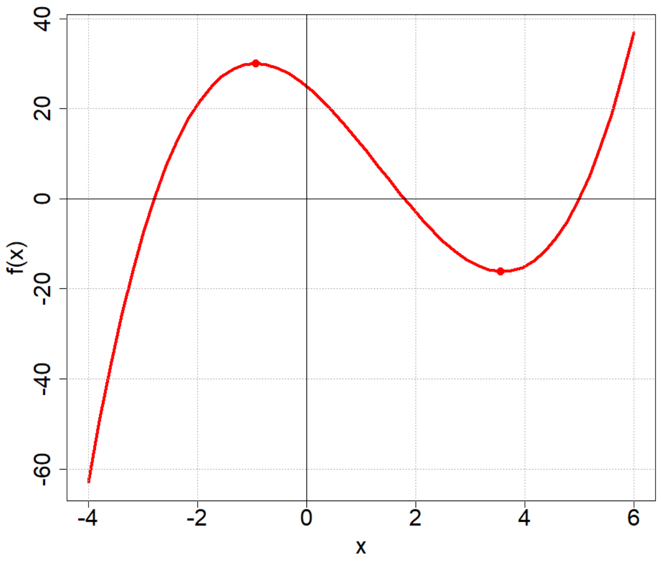
$$\frac{dy}{dx} < 0\text{: function is decreasing}$$

- What is happening when $\frac{dy}{dx} = 0$?
  - One possibility: function is “turning around”
    - » This is a "critical point"
  - Another possibility: inflection point. (consider $y = x^3$ at $x = 0$.)



Procedure for finding minimums and maximums

- Take first derivative
  - Where does first derivative equal zero?
  - These are candidate points for min or max ("critical points")
- Example: $f(x) = -x^2 + 5x - 2$
  - $\frac{df}{dx} = -2x + 5 := 0 \Rightarrow x = \frac{5}{2} = 2.5$
- Take second derivative
  - Use second derivatives to determine how the change is changing
  - Minimum: $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$
  - Maximum: $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$
  - See technical notes on next slide.
  - Ex: $\frac{d^2f}{d^2x} = -2 < 0 \Rightarrow$ Maxiumum



Finding minimums and maximums (technical notes)

- Technically, $\frac{dy}{dx} = 0$ is a *necessary* condition for a min or max. (In order to a point to be a min or max, $\frac{dy}{dx}$ must be zero.)
- $\frac{d^2y}{dx^2} > 0$ is a *sufficient* condition for a minimum, and $\frac{d^2y}{dx^2} < 0$ is a *sufficient* condition for a maximum.
  - But, these are not *necessary* conditions.
  - That is, there could be a minimum at a point where $\frac{d^2y}{dx^2} = 0$.
  - This is a technical detail that you almost certainly don't need to know until you take higher-level applied math.
  - For a good, quick review of *necessary* and *sufficient* conditions, watch this 3-minute video: https://www.khanacademy.org/partner-content/wi-phi/critical-thinking/v/necessary-sufficient-conditions

Inflection points

- Inflection point is where the function changes from concave to convex, or vice versa
- Second derivative tells us about concavity of the original function
- Inflection point: $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$
  - Technical: $\frac{d^2y}{dx^2} = 0$ is a necessary but not sufficient condition for inflection point
- That's enough for our purposes.
  - Just know that an inflection point is where $\frac{dy}{dx} = 0$ but the point is not a min or a max.
  - For more information I recommend:
    - https://www.mathsisfun.com/calculus/maxima-minima.html (easiest)
    - http://clas.sa.ucsb.edu/staff/lee/Max%20and%20Min's.htm
    - http://clas.sa.ucsb.edu/staff/lee/Inflection%20Points.htm
    - http://www.sosmath.com/calculus/diff/der13/der13.html
    - http://mathworld.wolfram.com/InflectionPoint.html (most technical)

Further resources

- These slides, notes, sample problems (see email)
- Strang textbook: http://ocw.mit.edu/resources/res-18-001-calculus-online-textbook-spring-2005/textbook/
- Strang videos at http://ocw.mit.edu/resources/res-18-005-highlights-of-calculus-spring-2010/ (see "highlights of calculus")
- Khan Academy videos: https://www.khanacademy.org/math
- Math(s) Is Fun: https://www.mathsisfun.com/links/index.html (10 upwards; algebra, calculus)
- Numerous other resources online. Find what works for you.
- Wolfram Alpha computational knowledge engine at http://www.wolframalpha.com/
  - Often useful for checking intuition or calculations
  - Excellent way to get a quick graph of a function