Calculus Review

Nicholas School of the Environment
Master's of Environmental Management Program

This set of lecture notes is designed only as a review of calculus. I assume that you have already been exposed to this material and simply need a quick refresher. For further review, I suggest the textbook and video series (especially the 5-video series "highlights of calculus") by MIT professor Gilbert Strang, which are available for free online at MIT open courseware. Here is the link:

http://ocw.mit.edu/ans7870/resources/Strang/strangtext.htm

1 Solving Systems of Linear Equations

In order to solve a system of linear equations you must have as many equations as you have variables (also called unknowns). If you have two variables you are trying to solve for you must have 2 equations. Three variables, three equations, and so on.

Example 1 *Solve for the market equilibrium*

$$\begin{aligned} Q^s &= P \\ Q^d &= 12 - 3P \end{aligned}$$

Technically we cannot yet solve this problem because we have three unknowns $\left(Q^s, Q^d, P\right)$ and only 2 equations. What we need to solve this problem is that in equilibrium, quantity supplied equals quantity demanded

$$Q^s = Q^d$$

Now we can solve by setting the two equations equal and solving for P, and then substituting the value we get for P into either of the two equations to solve for Q^s and Q^d.

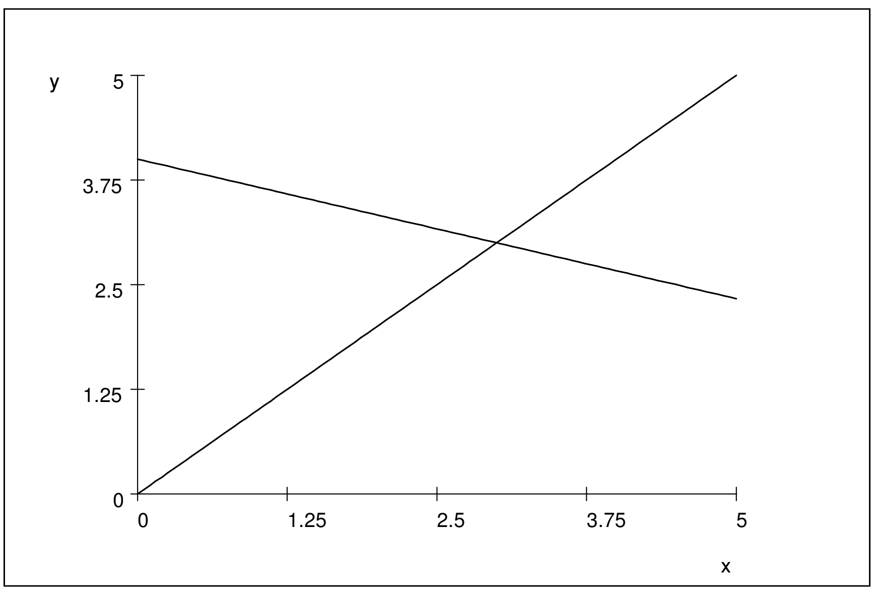
$$\begin{aligned} P &= 12 - 3P \\ 4P &= 12 \\ P &= 3 \\ Q^s &= Q^d = 3 \end{aligned}$$

We could have also solved this graphically. To do this we need to solve for P in terms of Q^s and Q^d and then graph.

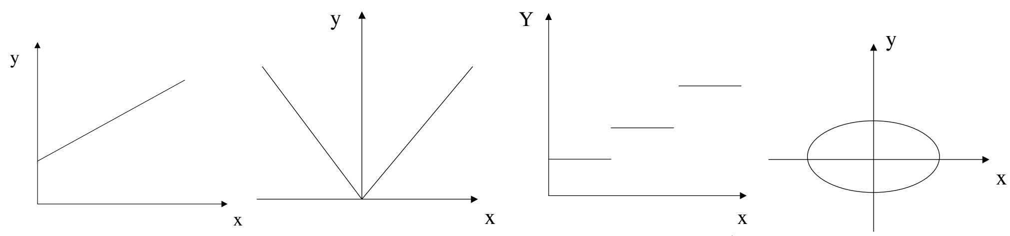
$$
\begin{aligned}
Q^s &= P \\
P &= Q^s
\end{aligned}
$$

$$
\begin{aligned}
Q^d &= 12 - 3P \\
Q^d - 12 &= -3P \\
P &= 4 - \frac{1}{3}Q^d
\end{aligned}
$$

Now we can graph both of these lines and see where they intersect. Supply will be an upward sloping line that intersects the Y-axis at 0 and has a slope of 1. Demand will be a downward sloping line that intersects the Y-axis at 4 and has a slope of $-\frac{1}{3}$. You can see that the demand and supply curves intersect at Q=3 and P=3.



2 Characteristics of Functions



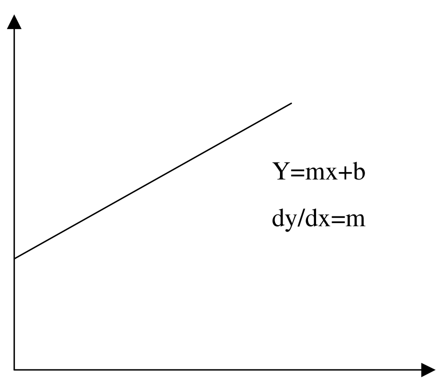
A **function** is a correspondence that maps each value of x (the independent variable) into a **unique** value of y (the dependent variable). Graphically it is easy to see if a correspondence is a function by seeing if for every x value there is one and only one value for y. The left three graphs above represent functions.

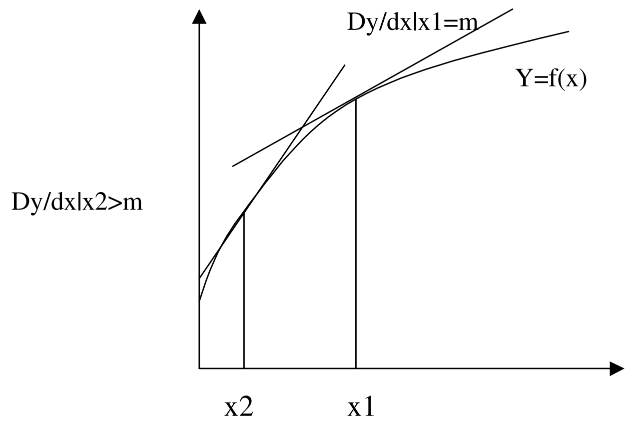
A **continuous function** is a function with no break in it. You can draw a continuous function without lifting your pencil. The left two graphs represent **continuous functions**.

A **continuously differentiable** function is a function where the first derivative exists for every value of x. Only the left most graph represents a continuously differentiable function.

3 Differentiation in one variable

The derivative of a function represents the rate of change of the function. For a linear function, the derivative is the slope of the line. For non-linear functions the derivative will not be constant, but rather will represent the slope of a tangent to the curve at a particular point.





In any case, the derivative can always be approximated by the change in y over the change in x. You might have seen this as "rise over run."

$$m = \frac{\Delta y}{\Delta x}$$

Now imagine that the change in y gets very very small. The limit as the change goes to zero is the derivative:

$$\frac{dy}{dx} = \lim_{\Delta \to 0} \frac{\Delta y}{\Delta x}$$

Rule 1: Power Rule

$$\begin{aligned} y &= kx^a \\ \frac{dy}{dx} &= akx^{a-1} \end{aligned}$$

Example 2

$$\begin{aligned} y &= 15x^3 \\ \frac{dy}{dx} &= 45x^2 \end{aligned}$$

Rule 2: Derivative of a constant

$$\begin{aligned} y &= k \\ \frac{dy}{dx} &= 0 \end{aligned}$$

Rule 3: Chain Rule

$$
\begin{aligned}
y &= f(g(x)) \\
\frac{dy}{dx} &= \frac{df}{dg}\frac{dg}{dx}
\end{aligned}
$$

Example 3

$$
\begin{aligned}
y &= (-3x+5)^2 \\
\frac{dy}{dx} &= 2(-3x+5)^1 * -3 \\
&= -6(-3x+15) \\
&= 18x-90
\end{aligned}
$$

Rule 4: Product Rule

$$
\begin{aligned}
y &= f(x)\,g(x) \\
\frac{dy}{dx} &= \frac{df}{dx}g(x) + \frac{dg}{dx}f(x)
\end{aligned}
$$

Example 4

$$
\begin{aligned}
y &= x^2(x-2) \\
\frac{dy}{dx} &= 2x(x-2) + 1\left(x^2\right) \\
&= 2x^2 - 4x + x^2 \\
&= 3x^2 - 4x
\end{aligned}
$$

Rule 5: Quotient Rule

$$
\begin{aligned}
y &= \frac{f(x)}{g(x)} \\
\frac{dy}{dx} &= \frac{\frac{df}{dx}g(x) - \frac{dg}{dx}f(x)}{g(x)^2}
\end{aligned}
$$

Example 5

$$
\begin{aligned}
y &= \frac{x^2}{(x-2)} \\
\frac{dy}{dx} &= \frac{2x(x-2) - 1\left(x^2\right)}{(x-2)^2} \\
&= \frac{2x^2 - 4x - x^2}{(x-2)^2} \\
&= \frac{x^2 - 4x}{(x-2)^2}
\end{aligned}
$$

4 Rules of Exponents and Logarithms

Rule 1: The natural log (ln) is the inverse of the exponential function e. The two functions essentially cancel each other out.

Example 6 $\ln\left(e^7\right) = 7$

Example 7 $e^{\ln(x)} = x$

Rule 2: The log of a product is the sum of the logs.

Example 8 $\ln(AB) = \ln(A) + \ln(B)$

Example 9 $\ln\left(Ae^7\right) = \ln(A) + \ln\left(e^7\right) = \ln(A) + 7$

Rule 3: The log of a quotient is the difference of the logs.

Example 10 $\ln\left(\frac{e^2}{c}\right) = \ln\left(e^2\right) - \ln(c) = 2 - \ln(c)$

Example 11 $\ln\left(\frac{e^2}{e^5}\right) = \ln\left(e^2\right) - \ln\left(e^5\right) = 2 - 5 = -3$

Rule 4: The log of a power equals the power time the log.

Example 12 $\ln\left(e^{15}\right) = 15 \ln e = 15$

Example 13 $\ln\left(A^3\right) = 3 \ln(A)$

Example 14 $\ln(uv^a) = \ln(u) + \ln(v^a) = \ln(u) + a \ln(v)$

Rule 5: Derivatives of $\ln(x)$

$$\begin{aligned} \frac{d\ln(x)}{dx} &= \frac{1}{x} \\ \frac{d\ln(2x)}{dx} &= \frac{1}{2x} * 2 = \frac{1}{x} \end{aligned}$$

Rule 6: Derivatives of e^x

$$\begin{aligned} \frac{de^x}{dx} &= e^x \\ \frac{de^{2x}}{dx} &= 2e^{2x} \end{aligned}$$

5 Partial Derivatives

A partial derivative represents the rate of change of a multivariable function along one variable's dimension, holding all the other variables constant. For example, if you have demand as a function of income and price, the partial derivative of demand with respect to price represents the small change in quantity demanded resulting from a small change in price, holding income constant.

Example 15

$$\begin{aligned} z &= 3x^2y^3 \\ \frac{\partial z}{\partial x} &= 6xy^3 \\ \frac{\partial z}{\partial y} &= 9x^2y^2 \end{aligned}$$

Example 16

$$\begin{aligned} f(x,y,z) &= xyz + x^3y + z^8 \\ \frac{\partial f}{\partial x} &= yz + 3x^2y \\ \frac{\partial f}{\partial y} &= xz + x^3 \\ \frac{\partial f}{\partial z} &= xy + 8z^7 \end{aligned}$$

Example 17

$$\begin{aligned} z &= 4x^2 + y^5 \\ \frac{\partial z}{\partial x} &= 8x \\ \frac{\partial z}{\partial y} &= 5y^4 \end{aligned}$$

6 The Total Derivative

The total derivative represents the change in a multivariate function with respect to all variables. It is the sum of the partial derivatives of a function for each variable multiplied by the change in that variable. In other words, if you have a function $F(x,y)$ its total derivative is:

$$\frac{\partial F(x,y)}{\partial x}dx + \frac{\partial F(x,y)}{\partial y}dy$$

Example 18

$$\begin{aligned} z &= 3x^2y^3 \\ \textit{Total Derivative} &= 6xy^3dx + 9x^2y^2dy \end{aligned}$$

7 Integration–The Indefinite Integral

Integration is the "reverse" of differentiation. The FUNDAMENTAL THEOREM OF CALCULUS is that the integral of the derivative is the original function plus some constant of integration.

$$\int \frac{df}{dx} dx = f(x) + c$$

We begin by talking about an indefinite integral. It is indefinite because taking the integral does not give you a specific value. Rather the result of the indefinite integral is a function. Integration essentially undoes differentiation, but a bit imperfectly because two functions that vary only by a constant have the same derivative. But when you integrate back up, you aren't sure what the original constant was. For example:

$$\begin{aligned} f(x) &= x^2 + 2 \\ \frac{df}{dx} &= 2x \end{aligned}$$

$$\begin{aligned} f(x) &= x^2 + 200 \\ \frac{df}{dx} &= 2x \end{aligned}$$

$$\int 2x dx = x^2 + c$$

Rule 1: The Power Rule

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + c$$

Example 19

$$\int x^3 dx = \frac{1}{4} x^4 + c$$

Rule 2: The Exponential Rule

$$\int e^x dx = e^x + c$$

Example 20

$$\int 2e^{2x} dx = e^{2x} + c$$

Rule 3: The Logarithmic Rule

$$\int \frac{1}{x} dx = \ln(x) + c$$

Example 21

$$\int \frac{2x}{x^2} dx = \ln(x^2) + c$$

Rule 4: Integrals of sums
The integral of a sum is the sum of the integrals

$$\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx$$

Example 22

$$\begin{aligned} & \int \left(x^3 + x^2 + 2x\right) dx \\ = & \int x^3 dx + \int x^2 dx + \int 2x dx \\ = & \frac{1}{4}x^4 + \frac{1}{3}x^3 + x^2 + c \end{aligned}$$

Rule 5: Integrals involving multiplication (basic version)
The integral of a constant times a function is the constant times the integral of the function.

$$\int kf(x)\, dx = k \int f(x)\, dx$$

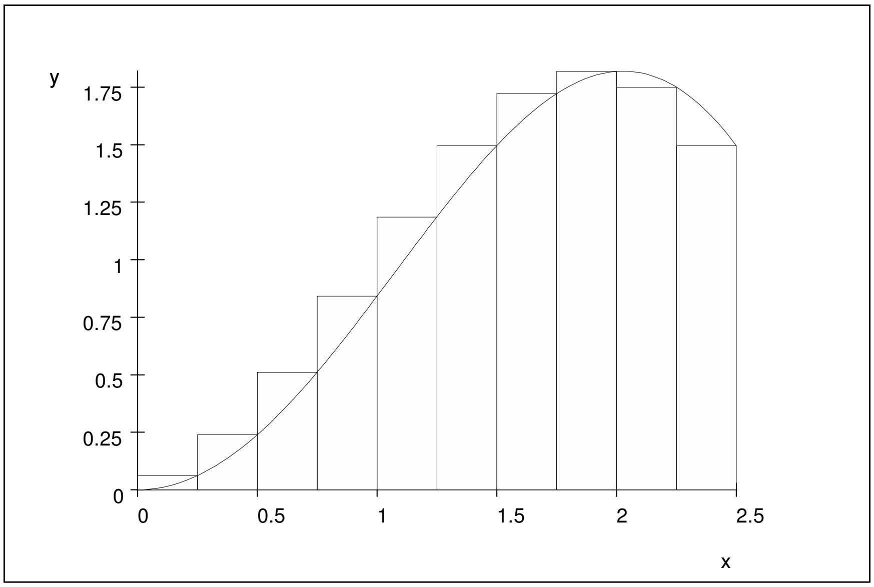
Example 23

$$\begin{aligned} \int 2x^2 dx &= 2\int x^2 dx \\ &= 2(\frac{1}{3}x^3) + c \\ &= \frac{2}{3}x^3 + c \end{aligned}$$

In your calculus class you likely spent a lot of time learning how to do very difficult indefinite integrals using substitution and integration by parts. I'm not going to review those methods due to time constraints, but you can find them in the online calculus text (Strang, see link at top) in chapters 5.4 and 7.1.

8 Integration–The Definite Integral

The definite integral represents the area under the curve between two points. The graph below is for the function $x \sin x$. Imagine that we wanted to know the area under that function from 0 to 2.5.



We can approximate the area under the curve by dividing the area into a series of rectangles and calculating the area for each rectangle and summing them up. If you divide the graph into n equal width rectangles, then the area under the curve from values a to b can be given by:

$$\frac{b-a}{n}\sum_{i=1}^{n} f\left(a+i\frac{b-a}{n}\right)$$

In our example, $n=10, b=2.5, a=0$. So the Riemann sum representing the area under the curve can be calculated as:

$$\begin{aligned} & 0.25\sum_{i=1}^{n} f\left(0.25i\right) = .25\sum_{i=1}^{n}\left(0.25i\right)\sin\left(0.25i\right) \\ = \; & 0.25\left(f\left(0.25\right)+f(0.5)+f\left(0.75\right)+f\left(1\right)+...+f\left(2.25\right)\right) \\ = \; & 2.781 \end{aligned}$$

If you let the width of the boxes get very very small you would improve your approximation of the area under the curve. The limit of the Riemann sum when n goes to infinity is the definite integral. It is written as follows:

$$\int_{a}^{b} f\left(x\right)dx$$

To calculate a definite integral, you take the indefinite integral (forgetting about the constant of integration) and evaluate that integral at the upper limit of integration (b) and then you evaluation the indefinite integral at the lower limit of integration (a) and subtract the second from the first.

Example 24

$$\begin{aligned} & \int_2^6 3x^2 dx \\ & \textit{First take the indefinite integral} \\ & x^3 + c \\ = & \textit{ Evaluate at 6 and forget } c \\ 6^3 = & \ 216 \\ & \textit{Evaluate at 2 and forget } c \\ 2^3 = & \ 8 \\ & \textit{Subtract latter from former} \\ 216 - 8 = & \ 208 \end{aligned}$$

How good was our Riemann sum estimate of the integral of $x \sin x$ from 0 to 2.5? That integral is fairly nasty to compute, but software can do it easily. The actual definite integral is:

$$\int_0^{2.5} x \sin x dx = 2.6013$$

and our Riemann sum estimate using 10 rectangles was 2.781.

9 Exponential Growth and Decay

Imagine that you \$100 to put in the bank and the interest rate (compounded annually) is 3%. If you make no further deposits or withdrawals, how much money will you have in 2 years? In 5 years?

Since the interest is compounded annually, after 1 year you have:

$$Y_1 = \$100\,(1 + 0.03) = \$103$$

After 2 years you have:

$$Y_2 = Y_1\,(1 + 0.03) = \$100\,(1 + 0.03)\,(1 + 0.03) = \$100\,(1 + 0.03)^2 = \$106.09$$

The general formula for calculating your balance with annually compounding interest is:

$$Y_t = Y_0\,(1 + r)^t$$

In this formula, Y denotes the amount of money, r denotes the interest rate, and t denotes the number of years. So in 5 years you have:

$$Y_5 = 100\,(1.03)^5 = \$115.93$$

What if the interest compounded more frequently? What if the interest compounded instantaneously? In this case you can replace the discrete growth equation with a continuous growth equation:

$$Y_t = Y_0 e^{rt}$$

Using this formulation after 2 years you have:

$$Y_2 = 100e^{0.03(2)} = 106.18$$

and after five years you have:

$$Y_5 = 100e^{0.03(5)} = \$116.18$$

Using the continuous time formula, you can also easily solve the following types of problems: In how many years will you have \$200 in the bank?

$$\begin{aligned} 200 &= 100e^{0.03t} \\ 2 &= e^{0.03t} \\ \ln(2) &= 0.03t \\ \frac{\ln(2)}{0.03} &= t \\ t &= 23.105 \end{aligned}$$

An interesting feature of these exponential growth/decay equations is that the amount of time it takes to go from \$100 to \$200 is the same amount of time it takes to go from \$200 to \$400. What matters is the amount of growth not the starting point. For those of you used to thinking about chemistry this is similar to the half-life of a decaying chemicals. It takes the same amount of time for a chemical to decay from 200 grams to 100 grams (decay by half) as it takes to go from 400 grams to 200 grams. The only difference in the chemistry example is that r is negative since the chemical is decaying rather than growing.

To get a better since of the relationship between Y_0 and Y_t, let's solve the problem one more time going from \$200 to \$400.

$$\begin{aligned} Y_t &= Y_o e^{rt} \\ 400 &= 200e^{0.03t} \\ 2 &= e^{0.03t} \\ \ln(2) &= 0.03t \\ \frac{\ln(2)}{0.03} &= t \\ t &= 23.105 \end{aligned}$$

The other interesting feature of the growth equation is how r affects the doubling (or half) time. Intuitively the higher the interest rate the sooner you should double your money. Just to see that this is, indeed, the case, let's solve the problem with the interest rate set to 6%. How long until we go from \$100 to \$200?

$$\begin{aligned}
200 &= 100e^{0.06t} \\
2 &= e^{0.06t} \\
\ln(2) &= 0.06t \\
\frac{\ln(2)}{0.06} &= t \\
t &= 11.552
\end{aligned}$$

So we doubled the interest rate and we cut the doubling time by half (from 23 years to 11.5 years).